

# CARTAN SUBALGEBRAS OF AMALGAMATED FREE PRODUCT $\text{II}_1$ FACTORS

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**ABSTRACT.** We study Cartan subalgebras in the context of amalgamated free product  $\text{II}_1$  factors and obtain several uniqueness and non-existence results. We prove that if  $\Gamma$  belongs to a large class of amalgamated free product groups (which contains the free product of any two infinite groups) then any  $\text{II}_1$  factor  $L^\infty(X) \rtimes \Gamma$  arising from a free ergodic probability measure preserving action of  $\Gamma$  has a unique Cartan subalgebra, up to unitary conjugacy. We also prove that if  $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$  is the free product of any two non-hyperfinite countable ergodic probability measure preserving equivalence relations, then the  $\text{II}_1$  factor  $L(\mathcal{R})$  has a unique Cartan subalgebra, up to unitary conjugacy. Finally, we show that the free product  $M = M_1 * M_2$  of any two  $\text{II}_1$  factors does not have a Cartan subalgebra. More generally, we prove that if  $A \subset M$  is a diffuse amenable von Neumann subalgebra and  $P \subset M$  denotes the algebra generated by its normalizer, then either  $P$  is amenable, or a corner of  $P$  can be unitarily conjugate into  $M_1$  or  $M_2$ .

## 1. INTRODUCTION

A *Cartan subalgebra* of a  $\text{II}_1$  factor  $M$  is a maximal abelian von Neumann subalgebra  $A$  whose normalizer generates  $M$ . The study of Cartan subalgebras plays a central role in the classification of  $\text{II}_1$  factors arising from probability measure preserving (pmp) actions. If  $\Gamma \curvearrowright (X, \mu)$  is a free ergodic pmp action of a countable group  $\Gamma$ , then the *group measure space*  $\text{II}_1$  factor  $L^\infty(X) \rtimes \Gamma$  [MvN36] contains  $L^\infty(X)$  as a Cartan subalgebra. In order to classify  $L^\infty(X) \rtimes \Gamma$  in terms of the action  $\Gamma \curvearrowright X$ , one would ideally aim to show that  $L^\infty(X)$  is its unique Cartan subalgebra (up to conjugation by an automorphism). Proving that certain classes of group measure space  $\text{II}_1$  factors have a unique Cartan subalgebra is useful because it reduces their classification, up to isomorphism, to the classification of the corresponding actions, up to orbit equivalence. Indeed, following [Si55, FM77], two free ergodic pmp actions  $\Gamma \curvearrowright X$  and  $\Lambda \curvearrowright Y$  are *orbit equivalent* if and only if there exists an isomorphism  $\theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$  such that  $\theta(L^\infty(X)) = L^\infty(Y)$ .

In the case of  $\text{II}_1$  factors coming from actions of amenable groups, both the classification and uniqueness of Cartan problems have been completely settled since the early 1980's. A celebrated theorem of A. Connes [Co76] asserts that all  $\text{II}_1$  factors arising from free ergodic pmp actions of infinite amenable groups are isomorphic to the hyperfinite  $\text{II}_1$  factor,  $R$ . Additionally, [CFW81] shows that any two Cartan subalgebras of  $R$  are conjugate by an automorphism of  $R$ .

For a long time, however, the questions of classification and uniqueness of Cartan subalgebras for  $\text{II}_1$  factors associated with actions of non-amenable groups, were considered intractable. During the last decade, S. Popa's *deformation/rigidity* theory has led to spectacular progress in the classification of group measure space  $\text{II}_1$  factors (see the surveys [Po07, Va10a, Io12]). This was in part made possible by several results providing classes of group measure space  $\text{II}_1$  factors that have a unique Cartan subalgebra, up to unitary conjugacy. The first such classes were obtained by N. Ozawa and S. Popa in their breakthrough work [OP07, OP08]. They showed that  $\text{II}_1$  factors  $L^\infty(X) \rtimes \Gamma$  associated with free ergodic *profinite* actions of free groups  $\Gamma = \mathbb{F}_n$  and their direct products  $\Gamma = \mathbb{F}_{n_1} \times \mathbb{F}_{n_2} \times \dots \times \mathbb{F}_{n_k}$  have a unique Cartan subalgebra, up to unitary conjugacy. Recently, this result has been extended to profinite actions of hyperbolic groups [CS11] and of

direct products of hyperbolic groups [CSU11]. The proofs of these results rely both on the fact that free groups (and, more generally, hyperbolic groups, see [Oz07], [Oz10]) are *weakly amenable* and that the actions are profinite.

In a very recent breakthrough, S. Popa and S. Vaes succeeded in removing the profiniteness assumption on the action and obtained wide-ranging unique Cartan subalgebra results. They proved that if  $\Gamma$  is either a weakly amenable group with  $\beta_1^{(2)}(\Gamma) > 0$  [PV11] or a hyperbolic group [PV12] (or a direct product of groups in one of these classes), then  $\text{II}_1$  factors  $L^\infty(X) \rtimes \Gamma$  arising from *arbitrary* free ergodic pmp actions of  $\Gamma$  have a unique Cartan subalgebra, up to unitary conjugacy. Following [PV11, Definition 1.4], such groups  $\Gamma$ , whose every action gives rise to a  $\text{II}_1$  factor with a unique Cartan subalgebra, are called  $\mathcal{C}$ -*rigid* (Cartan rigid).

In this paper we study Cartan subalgebras of tracial amalgamated free product von Neumann algebras  $M = M_1 *_B M_2$  (see [Po93, VDN92] for the definition). Our methods are best suited to the case when  $M = L^\infty(X) \rtimes \Gamma$  comes from an action of an amalgamated free product group  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$ . In this context, by imposing that the inclusion  $\Lambda < \Gamma$  satisfies a weak malnormality condition [PV09], we prove that  $L^\infty(X)$  is the unique Cartan subalgebra of  $M$ , up to unitary conjugacy, for *any* free ergodic pmp action  $\Gamma \curvearrowright X$ .

**Theorem 1.1.** *Let  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  be an amalgamated free product group such that  $[\Gamma_1 : \Lambda] \geq 2$  and  $[\Gamma_2 : \Lambda] \geq 3$ . Assume that there exist  $g_1, g_2, \dots, g_n \in \Gamma$  such that  $\cap_{i=1}^n g_i \Lambda g_i^{-1}$  is finite. Let  $\Gamma \curvearrowright (X, \mu)$  be any free ergodic pmp action of  $\Gamma$  on a standard probability space  $(X, \mu)$ .*

*Then the  $\text{II}_1$  factor  $M = L^\infty(X) \rtimes \Gamma$  has a unique Cartan subalgebra, up to unitary conjugacy.*

*Moreover, the same holds if  $\Gamma$  is replaced with a direct product of finitely many such groups  $\Gamma$ .*

This result provides the first examples of  $\mathcal{C}$ -rigid groups  $\Gamma$  that are not weakly amenable (take e.g.  $\Gamma = SL_3(\mathbb{Z}) * \Sigma$ , where  $\Sigma$  is any non-trivial countable group).

Theorem 1.1 generalizes and strengthens the main result of [PV09]. Indeed, in the above setting, assume further that  $\Lambda$  is amenable and that  $\Gamma_2$  contains either a non-amenable subgroup with the relative property (T) or two non-amenable commuting subgroups. [PV09, Theorem 1.1] then asserts that  $M$  has a unique *group measure space* Cartan subalgebra.

Theorem 1.1 provides strong supporting evidence for a general conjecture which predicts that any group  $\Gamma$  with positive first  $\ell^2$ -Betti number,  $\beta_1^{(2)}(\Gamma) > 0$ , is  $\mathcal{C}$ -rigid. Thus, it implies that the free product  $\Gamma = \Gamma_1 * \Gamma_2$  of any two countable groups satisfying  $|\Gamma_1| \geq 2$  and  $|\Gamma_2| \geq 3$ , is  $\mathcal{C}$ -rigid.

Recently, there have been several results offering positive evidence for this conjecture. Firstly, it was shown in [PV09] that if  $\Gamma = \Gamma_1 * \Gamma_2$ , where  $\Gamma_1$  is a property (T) group and  $\Gamma_2$  is a non-trivial group, then any  $\text{II}_1$  factor  $L^\infty(X) \rtimes \Gamma$  associated with a free ergodic pmp action of  $\Gamma$  has a unique group measure space Cartan subalgebra, up to unitary conjugacy (see also [FV10, HPV10]).

Secondly, the same has been proven in [CP10] under the assumption that  $\beta_1^{(2)}(\Gamma) > 0$  and  $\Gamma$  admits a non-amenable subgroup with the relative property (T). For a common generalization of the last two results, see [Va10b]. Thirdly, we proved that if  $\beta_1^{(2)}(\Gamma) > 0$ , then  $L^\infty(X) \rtimes \Gamma$  has a unique group measure space Cartan subalgebra whenever the action  $\Gamma \curvearrowright (X, \mu)$  is either rigid [Io11a] or compact [Io11b]. As already mentioned above, the conjecture has been very recently established in full generality for weakly amenable groups  $\Gamma$  with  $\beta_1^{(2)}(\Gamma) > 0$  in [PV11].

As a consequence of Theorem 1.1 we obtain a new family of  $W^*$ -superrigid actions. Recall that a free ergodic pmp action  $\Gamma \curvearrowright (X, \mu)$  is called  $W^*$ -*superrigid* if whenever  $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ , for some free ergodic pmp action  $\Lambda \curvearrowright (Y, \nu)$ , the groups  $\Gamma$  and  $\Lambda$  are isomorphic, and their actions are conjugate. The existence of virtually  $W^*$ -superrigid actions was proven in [Pe09]. The first

concrete families of  $W^*$ -superrigid actions were found in [PV09] where it was shown for instance that Bernoulli actions of many amalgamated free product groups have this property. In [Io10] we proved that Bernoulli actions of icc property (T) groups are  $W^*$ -superrigid. By combining Theorem 1.1 with the cocycle superrigidity theorem [Po06a] we derive the following.

**Corollary 1.2.** *Let  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  and  $\Gamma' = \Gamma'_1 *_Lambda \Gamma'_2$  be two amalgamated free product groups satisfying the hypothesis of Theorem 1.1. Denote  $G = \Gamma \times \Gamma'$ .*

*Then any free action of  $G$  which is a quotient of the Bernoulli action  $G \curvearrowright [0, 1]^G$  is  $W^*$ -superrigid.*

Next, we return to the study of Cartan subalgebras of general amalgamated free product  $\text{II}_1$  factors  $M = M_1 *_B M_2$ . Assuming that  $B$  is amenable and  $M$  satisfies some rather mild conditions, we prove that any Cartan subalgebra  $A \subset M$  has a corner which embeds into  $B$ , in the sense of S. Popa's *intertwining-by-bimodules* [Po03] (see Theorem 2.1). This condition, written in symbols as  $A \prec_M B$ , roughly means that  $A$  can be conjugated into  $B$  via a unitary element from  $M$ .

**Theorem 1.3.** *Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be two tracial von Neumann algebras with a common amenable von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B$ . Assume that  $M = M_1 *_B M_2$  is a factor and that either:*

- (1)  $M_1$  and  $M_2$  have no amenable direct summands, or
- (2)  $M$  does not have property  $\Gamma$  and  $pM_1p \neq pMp \neq pM_2p$ , for any non-zero projection  $p \in B$ .

*If  $A \subset M$  is a Cartan subalgebra, then  $A \prec_M B$ .*

Recall that a *tracial von Neumann algebra*  $(M, \tau)$  is a von Neumann algebra  $M$  endowed with a normal faithful tracial state  $\tau$ . As usual, we denote by  $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$  the induced Hilbert norm on  $M$ . Recall also that a  $\text{II}_1$  factor  $M$  has *property  $\Gamma$*  if there exists a sequence  $u_n \in M$  of unitary elements such that  $\tau(u_n) = 0$ , for all  $n$ , and  $\|u_n x - x u_n\|_2 \rightarrow 0$ , for every  $x \in M$  [MvN43].

Theorem 1.3 has two interesting applications.

Firstly, it yields a classification result for von Neumann algebras  $L(\mathcal{R})$  [FM77] arising from the *free product*  $\mathcal{R} = \mathcal{R}_1 *_\mathcal{R}_2$  of two equivalence relations (see [Ga99] for the definition). For instance, it implies that if  $\mathcal{R}_1, \mathcal{R}_2$  are ergodic and non-hyperfinite, then any countable pmp equivalence relation  $\mathcal{S}$  such that  $L(\mathcal{S}) \cong L(\mathcal{R})$  is necessarily isomorphic to  $\mathcal{R}$ . More generally, we have

**Corollary 1.4.** *Let  $\mathcal{R}$  be a countable ergodic pmp equivalence relation on a standard probability space  $(X, \mu)$ . Assume that  $\mathcal{R} = \mathcal{R}_1 *_\mathcal{R}_2$ , for two equivalence relations  $\mathcal{R}_1$  and  $\mathcal{R}_2$  on  $(X, \mu)$ . Additionally, suppose that either:*

- (1)  $\mathcal{R}_1|_Y$  and  $\mathcal{R}_2|_Y$  are not hyperfinite, for any Borel set  $Y \subset X$  with  $\mu(Y) > 0$ , or
- (2)  $\mathcal{R}$  is strongly ergodic, and  $\mathcal{R}_1$  and  $\mathcal{R}_2$  have infinite orbits, almost everywhere.

*Then  $L^\infty(X)$  is the unique Cartan subalgebra of  $L(\mathcal{R})$ , up to unitary conjugacy.*

*Thus, if  $L(\mathcal{R}) \cong L(\mathcal{S})$ , for any ergodic countable pmp equivalence relation  $\mathcal{S}$ , then  $\mathcal{R} \cong \mathcal{S}$ .*

Here,  $\mathcal{R}|_Y := \mathcal{R} \cap (Y \times Y)$  denotes the restriction of  $\mathcal{R}$  to  $Y$ . Recall that an ergodic countable pmp equivalence relation  $\mathcal{R}$  on a probability space  $(X, \mu)$  is called *strongly ergodic* if there does not exist a sequence of Borel sets  $Y_n \subset X$  such that  $\mu(Y_n) = \frac{1}{2}$ , for all  $n$ , and  $\mu(\theta(Y_n) \Delta Y_n) \rightarrow 0$ , for any Borel automorphism  $\theta$  of  $X$  satisfying  $(\theta(x), x) \in \mathcal{R}$ , for almost every  $x \in X$ .

Secondly, Theorem 1.3 allows us to show that the free product of any two diffuse tracial von Neumann algebras does not have a Cartan subalgebra. By using the notion of free entropy for von Neumann algebras, D. Voiculescu proved that the free group factors  $L(\mathbb{F}_n)$  do not have

Cartan subalgebras [Vo95]. This result was extended in [Ju05, Lemma 3.7] to show that the free product  $M = M_1 * M_2$  of any two diffuse tracial von Neumann algebras  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$ , which are embeddable into  $R^\omega$ , does not have a Cartan subalgebra. Here we prove this result without requiring that  $M_1$  and  $M_2$  embed into  $R^\omega$ . More generally, we have

**Corollary 1.5.** *Let  $(M_1, \tau_1), (M_2, \tau_2)$  be tracial von Neumann algebras satisfying  $M_1 \neq \mathbb{C}1 \neq M_2$  and  $\dim(M_1) + \dim(M_2) \geq 5$ .*

*Then their free product  $M = M_1 * M_2$  does not have a Cartan subalgebra.*

Corollary 1.5 shows that if  $M_1 \neq \mathbb{C}1 \neq M_2$  and  $(\dim(M_1), \dim(M_2)) \neq (2, 2)$ , then  $M$  has no Cartan subalgebra. On the other hand, if  $\dim(M_1) = \dim(M_2) = 2$ , then  $M$  is of type I (see [Dy93, Theorem 1.1]) and therefore has a Cartan subalgebra.

So far, our results only apply to Cartan subalgebras of amalgamated free product von Neumann algebras  $M = M_1 *_B M_2$ . From now on, we more generally study, in the spirit of [OP07] and [PV11], normalizers of arbitrary diffuse amenable von Neumann subalgebras  $A \subset M$ . Recall that the *normalizer* of  $A$  in  $M$ , denoted  $\mathcal{N}_M(A)$ , is the group of unitaries  $u \in M$  such that  $uAu^* = A$ . Assuming that the normalizer of  $A$  satisfies a certain spectral gap condition, we prove the following dichotomy: either a corner of  $A$  embeds into  $M_i$ , for some  $i \in \{1, 2\}$ , or the algebra generated by the normalizer of  $A$  is amenable relative to  $B$ . More precisely, we show

**Theorem 1.6.** *Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be two tracial von Neumann algebras with a common von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B$ . Let  $M = M_1 *_B M_2$  and  $A \subset M$  be a von Neumann subalgebra which is amenable relative to  $B$ . Denote by  $P = \mathcal{N}_M(A)''$  the von Neumann algebra generated by the normalizer of  $A$  in  $M$ . Assume that  $P' \cap M^\omega = \mathbb{C}1$ , for a free ultrafilter  $\omega$  on  $\mathbb{N}$ .*

*Then one of the following conditions holds true:*

- (1)  $A \prec_M B$ .
- (2)  $P \prec_M M_i$ , for some  $i \in \{1, 2\}$ .
- (3)  $P$  is amenable relative to  $B$ .

For the definition of *relative amenability*, see Section 2.2. For now, note that if  $B$  is amenable, then  $P$  is amenable relative to  $B$  if and only if  $P$  is amenable. By a result of A. Connes [Co76], the condition  $P' \cap M^\omega = \mathbb{C}1$  holds if and only if the representation  $\mathcal{U}(P) \curvearrowright L^2(M) \ominus \mathbb{C}1$  given by conjugation has *spectral gap* (i.e. has no almost invariant vectors).

We believe that Theorem 1.6 should hold without assuming that  $P' \cap M^\omega = \mathbb{C}1$ , but we were unable to prove this for general  $B$ . Nevertheless, in the case  $B = \mathbb{C}$ , a detailed analysis of the relative commutant  $P' \cap M^\omega$  (see Section 6) enabled us to show that the condition  $P' \cap M^\omega = \mathbb{C}1$  is indeed redundant.

**Corollary 1.7.** *Let  $(M_1, \tau_1), (M_2, \tau_2)$  be two tracial von Neumann algebras. Let  $M = M_1 * M_2$  and  $A \subset M$  be a diffuse amenable von Neumann subalgebra. Denote  $P = \mathcal{N}_M(A)''$ .*

*Then either  $P \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or  $P$  is amenable.*

For a more precise version of this result in the case  $M_1$  and  $M_2$  are  $\text{II}_1$  factors, see Corollary 9.1.

Finally, we present a new class of strongly solid von Neumann algebras. Recall that a von Neumann algebra  $M$  is called *strongly solid* if  $\mathcal{N}_M(A)''$  is amenable, whenever  $A \subset M$  is a diffuse amenable von Neumann subalgebra [OP07]. N. Ozawa and S. Popa proved in [OP07] that the free group factors  $L(\mathbb{F}_n)$  are strongly solid. More generally, I. Chifan and T. Sinclair recently showed that the von Neumann algebra  $L(\Gamma)$  of any icc hyperbolic group  $\Gamma$  is strongly solid [CS11].

The class of strongly solid von Neumann algebras is not closed under taking amalgamated free products. For instance, if  $\mathbb{F}_2 \curvearrowright (X, \mu)$  is a pmp action on a non-atomic probability space  $(X, \mu)$ , then the group measure space algebra  $L^\infty(X) \rtimes \mathbb{F}_2 = (L^\infty(X) \rtimes \mathbb{Z}) *_{{L^\infty(X)}} (L^\infty(X) \rtimes \mathbb{Z})$  is not strongly solid, although the algebras involved in its amalgamated free product decomposition are amenable and hence strongly solid.

However, as an application of Theorem 1.6, we prove that the class of solid von Neumann algebras is closed under free products (Corollary 9.6). More generally, we show that if  $M_1$  and  $M_2$  are strongly solid von Neumann algebras, then the amalgamated free product  $M = M_1 *_B M_2$  is strongly solid, provided that the inclusions  $B \subset M_1$  and  $B \subset M_2$  are *mixing*, and  $B$  is amenable.

**Theorem 1.8.** *Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be strongly solid von Neumann algebras with a common amenable von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B$ . Assume that the inclusions  $B \subset M_1$  and  $B \subset M_2$  are mixing. Denote  $M = M_1 *_B M_2$ .*

*Then  $M$  is strongly solid.*

For the definition of mixing inclusions of von Neumann algebras, see Section 9.4. For now, let us point out that the inclusion  $B \subset M$  is mixing whenever the  $B$ - $B$  bimodule  $L^2(M) \ominus L^2(B)$  is contained in a multiple of the coarse  $B$ - $B$  bimodule  $L^2(B) \otimes L^2(B)$ .

Theorem 1.8 implies that if  $M_1, M_2, \dots, M_n$  are amenable von Neumann algebras with a common von Neumann subalgebra  $B$  such that the inclusions  $B \subset M_1, B \subset M_2, \dots, B \subset M_n$  are mixing, then  $M = M_1 *_B M_2 *_B \dots *_B M_n$  is strongly solid (Corollary 9.7).

**Comments on the proofs.** The most general type of result that we prove is Theorem 1.6. Let us say a few words about its proof. Assume therefore that  $A$  is a von Neumann subalgebra of an amalgamated free product von Neumann algebra  $M = M_1 *_B M_2$  that is amenable relative to  $B$ . We denote  $P = \mathcal{N}_M(A)''$  and assume that  $P' \cap M^\omega = \mathbb{C}1$ .

Our goal is to show that either  $A \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or  $P$  is amenable relative to  $B$ . This is enough to deduce the conclusion of Theorem 1.6, because by [IPP05, Theorem 1.1] the first case implies that either  $A \prec_M B$  or  $P \prec_M M_i$ , for some  $i \in \{1, 2\}$ .

The strategy of proof is motivated by a beautiful recent dichotomy theorem due to S. Popa and S. Vaes. To state the particular case of [PV11, Theorem 1.6] that will be useful to us, let  $\mathbb{F}_2 \curvearrowright (N, \tau)$  be a trace preserving action of the free group  $\mathbb{F}_2$  on a tracial von Neumann algebra  $(N, \tau)$ . Denote  $\tilde{M} = N \rtimes \mathbb{F}_2$ . Given a von Neumann subalgebra  $D \subset \tilde{M}$  that is amenable relative to  $N$ , it is shown in [PV11] that either  $D \prec_{\tilde{M}} N$  or  $\mathcal{N}_{\tilde{M}}(D)''$  is amenable relative to  $N$ .

In order to apply this result in our context, we use the *free malleable deformation* introduced in [IPP05]. More precisely, define  $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{F}_2))$ . Then  $M \subset \tilde{M}$  and one constructs a 1-parameter group of automorphisms  $\{\theta_t\}_{t \in \mathbb{R}}$  of  $\tilde{M}$  as follows. Let  $u_1, u_2 \in L(\mathbb{F}_2)$  be the canonical generating unitaries and  $h_1, h_2 \in L(\mathbb{F}_2)$  be hermitian elements such that  $u_1 = \exp(ih_1)$  and  $u_2 = \exp(ih_2)$ . For  $t \in \mathbb{R}$ , define the unitary elements  $u_1^t = \exp(it h_1)$  and  $u_2^t = \exp(it h_2)$ . Then there exists an automorphism  $\theta_t$  of  $\tilde{M}$  such that

$$\theta_t|_{M_1} = \text{Ad}(u_1^t)|_{M_1}, \quad \theta_t|_{M_2} = \text{Ad}(u_2^t)|_{M_2} \quad \text{and} \quad \theta_t|_{L(\mathbb{F}_2)} = \text{id}_{L(\mathbb{F}_2)}.$$

The starting point of the proof is the key observation that  $\tilde{M}$  can be written as  $\tilde{M} = N \rtimes \mathbb{F}_2$ , where  $N$  is the von Neumann subalgebra of  $\tilde{M}$  generated by  $\{u_g M u_g^*\}_{g \in \mathbb{F}_2}$  and  $\mathbb{F}_2$  acts on  $N$  via conjugation with  $\{u_g\}_{g \in \mathbb{F}_2}$ .

Now, let  $t \in (0, 1)$  and notice that  $\theta_t(P) \subset \mathcal{N}_{\tilde{M}}(\theta_t(A))''$ . Since  $A$  is amenable relative to  $B$  and  $\theta_t(B) = B \subset N$ , we deduce that  $\theta_t(A)$  is amenable relative to  $N$ . By applying the dichotomy

of [PV11], we conclude that either  $\theta_t(A) \prec_{\tilde{M}} N$  or  $\theta_t(P)$  is amenable relative to  $N$ . Since  $t \in (0, 1)$  is arbitrary, we are therefore in one of the following two cases:

- (1)  $\theta_t(A) \prec_{\tilde{M}} N$ , for some  $t \in (0, 1)$ .
- (2)  $\theta_t(P)$  is amenable relative to  $N$ , for any  $t \in (0, 1)$ .

The core of the paper consists of analyzing what can be said about the von Neumann subalgebras  $A$  and  $P$  of  $M$  which satisfy these conditions. Note that since  $\theta_1(M) \subset N$ , these conditions are trivially satisfied for any subalgebra  $A \subset M$  when  $t = 1$ .

Thus, we prove in Section 3 that if (1) holds then  $A \prec_M M_i$ , for some  $i \in \{1, 2\}$ . The proof of this result has two main ingredients. To explain what they are, assume by contradiction that  $A \not\prec_M M_i$ , for any  $i \in \{1, 2\}$ . Then [IPP05, Theorem 3.1] provides a sequence of unitary elements  $u_k \in A$  which are asymptotically (i.e., as  $k \rightarrow \infty$ ) supported on words in  $M_1 \ominus B$  and  $M_2 \ominus B$  of length  $\geq \ell$ , for every  $\ell \geq 1$ . In the second part of the proof, we use a calculation from the theory of random walks on groups to derive that the unitaries  $\theta_t(u_k) \in \theta_t(A)$  are asymptotically perpendicular to  $aNb$ , for any  $a, b \in \tilde{M}$ . This contradicts the assumption that (1) holds.

In Sections 4 and 5 we investigate which von Neumann subalgebras  $P \subset M$  satisfy (2). Our first result addressing this question asserts that if (2) holds for  $P = M$ , then  $M_1$  or  $M_2$  must have a amenable direct summand (see Theorem 4.1). In combination with the above, it follows that if  $A \subset M$  is a Cartan subalgebra, then either  $A \prec_M M_i$  or  $M_i$  has an amenable direct summand, for some  $i \in \{1, 2\}$ . This readily implies Theorem 1.3 under the first set of conditions.

However, in order to prove Theorem 1.6, we need consider arbitrary von Neumann subalgebras  $P \subset M$  which satisfy (2) in addition to the initial assumption that  $P' \cap M^\omega = \mathbb{C}1$ . Under these assumptions, we prove that either  $P \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or  $P$  is amenable relative to  $B$  (see Theorem 5.1). It is clear that this result completes the proof of Theorem 1.6.

**Organization of the paper.** Besides the introduction this paper has eight other sections. In Section 2 we recall the tools that are needed in the sequel as well as establish some new results. For instance, we prove that if  $A \subset M = M_1 *_B M_2$  is a von Neumann subalgebra that is amenable relative to  $M_1$ , then either  $A$  is amenable relative to  $B$ , or a corner of  $\mathcal{N}_M(A)''$  embeds into  $M_1$  (see Corollary 2.12). We have described above the contents of Section 3-5. In Section 6, motivated by the hypothesis of Theorem 1.6, we study the relative commutant  $P' \cap M^\omega$ , where  $P$  is a von Neumann subalgebra of an amalgamated free product algebra  $M = M_1 *_B M_2$ . Finally, Sections 7-9 are devoted to the proofs of the results stated in the introduction.

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## 2. PRELIMINARIES

We start by recalling some of the terminology that we use in this paper.

Throughout we work with *tracial* von Neumann algebras  $(M, \tau)$ , i.e. von Neumann algebras  $M$  endowed with a faithful, normal, tracial state  $\tau$ . We assume that  $M$  is *separable*, unless it is an ultraproduct algebra or we specify otherwise.

We denote by  $\mathcal{Z}(M)$  the *center* of  $M$ , by  $\mathcal{U}(M)$  the *group of unitaries* of  $M$  and by  $(M)_1$  the *unit ball* of  $M$ . We say that a von Neumann subalgebra  $A \subset M$  is *regular* in  $M$  if  $\mathcal{N}_M(A)'' = M$ .

For a free ultrafilter  $\omega$  on  $\mathbb{N}$ , the *ultraproduct* algebra  $M^\omega$  is defined as the quotient  $\ell^\infty(\mathbb{N}, M)/\mathcal{I}$ , where  $\mathcal{I} \subset \ell^\infty(\mathbb{N}, M)$  is the closed ideal of  $x = (x_n)_n$  such that  $\lim_{n \rightarrow \omega} \|x_n\|_2 = 0$ . As it turns out,  $M^\omega$  is a tracial von Neumann algebra, with its canonical trace given by  $\tau_\omega((x_n)_n) = \lim_{n \rightarrow \omega} \tau(x_n)$ .

If  $M$  and  $N$  are tracial von Neumann algebras, then an  $M$ - $N$  *bimodule* is a Hilbert space  $\mathcal{H}$  endowed with commuting normal  $*$ -homomorphisms  $\pi : M \rightarrow \mathbb{B}(\mathcal{H})$  and  $\rho : N^{\text{op}} \rightarrow \mathbb{B}(\mathcal{H})$ . For  $x \in M, y \in N$  and  $\xi \in \mathcal{H}$  we denote  $x\xi y = \pi(x)\rho(y)(\xi)$ .

Next, let  $M, N, P$  be tracial von Neumann algebras. Let  $\mathcal{H}$  and  $\mathcal{K}$  be  $M$ - $N$  and  $N$ - $P$  bimodules. Let  $\mathcal{K}_0$  be vector subspace of vectors  $\eta \in \mathcal{K}$  that are left bounded, i.e. for which there exists  $c > 0$  such that  $\|x\eta\| \leq c\|x\|_2$ , for all  $x \in N$ . The *Connes tensor product*  $\mathcal{H} \otimes_N \mathcal{K}$  is defined as the separation/completion of the algebraic tensor product  $\mathcal{H} \otimes \mathcal{K}_0$  with respect to the scalar product  $\langle \xi \otimes_N \eta, \xi' \otimes_N \eta' \rangle = \langle \xi y, \xi' \rangle$ , where  $y \in N$  satisfies  $\langle x\eta, \eta' \rangle = \tau(xy)$ , for all  $x \in N$ . Note that  $\mathcal{H} \otimes_N \mathcal{K}$  carries a  $M$ - $P$  bimodule structure given by  $x(\xi \otimes_N \eta)y = x\xi \otimes_N \eta y$ .

In the following six subsections we present the tools we will use in the proofs of our main results.

**2.1. Intertwining-by-bimodules.** We first recall from [Po03, Theorem 2.1 and Corollary 2.3] S. Popa's powerful *intertwining-by-bimodules* technique.

**Theorem 2.1.** [Po03] *Let  $(M, \tau)$  be a tracial von Neumann algebra and  $P, Q \subset M$  be two (not necessarily unital) von Neumann subalgebras. Then the following are equivalent:*

- *There exist non-zero projections  $p \in P, q \in Q$ , a  $*$ -homomorphism  $\phi : pPp \rightarrow qQq$  and a non-zero partial isometry  $v \in qMp$  such that  $\phi(x)v = vx$ , for all  $x \in pPp$ .*
- *There is no sequence  $u_n \in \mathcal{U}(P)$  satisfying  $\|E_Q(xu_n y)\|_2 \rightarrow 0$ , for all  $x, y \in M$ .*

*If one of these conditions holds true, then we say that a corner of  $P$  embeds into  $Q$  inside  $M$  and write  $P \prec_M Q$ .*

Note that if  $M$  is not separable, then the same statement holds if the sequence  $\{u_n\}_n$  is replaced by a net.

**2.2. Relative amenability.** A tracial von Neumann algebra  $(M, \tau)$  is called *amenable* if there exists a net  $\xi_n \in L^2(M) \bar{\otimes} L^2(M)$  such that  $\langle x\xi_n, \xi_n \rangle \rightarrow \tau(x)$  and  $\|x\xi_n - \xi_n x\|_2 \rightarrow 0$ , for every  $x \in M$ . By A. Connes' theorem [Co76],  $M$  is amenable iff it is approximately finite dimensional, i.e.  $M = (\cup_{n \geq 1} M_n)''$ , for an increasing sequence  $(M_n)_n$  of finite dimensional subalgebras of  $M$ .

Let  $Q \subset M$  be a von Neumann subalgebra. *Jones' basic construction*  $\langle M, e_Q \rangle$  is defined as the von Neumann subalgebra of  $\mathbb{B}(L^2(M))$  generated by  $M$  and the orthogonal projection  $e_Q$  from  $L^2(M)$  onto  $L^2(Q)$ . Recall that  $\langle M, e_Q \rangle$  has a faithful semi-finite trace given by  $\text{Tr}(xe_Q yL) = \tau(xy)$  for all  $x, y \in M$ . We denote by  $L^2(\langle M, e_Q \rangle)$  the associated Hilbert space and endow it with the natural  $M$ -bimodule structure. Note that  $L^2(\langle M, e_Q \rangle) \cong L^2(M) \otimes_Q L^2(M)$ , as  $M$ - $M$  bimodules.

Now, let  $P \subset pMp$  be a von Neumann subalgebra, for some projection  $p \in M$ . Following [OP07, Definition 2.2] we say that  $P$  is *amenable relative to  $Q$  inside  $M$*  if there exists a net  $\xi_n \in L^2(p\langle M, e_Q \rangle p)$  such that  $\langle x\xi_n, \xi_n \rangle \rightarrow \tau(x)$ , for every  $x \in pMp$ , and  $\|y\xi_n - \xi_n y\|_2 \rightarrow 0$ , for every  $y \in P$ . Note that when  $Q$  is amenable, this condition is equivalent to  $P$  being amenable.

By [OP07, Theorem 2.1], relative amenability is equivalent to the existence of a  $P$ -central state  $\phi$  on  $p\langle M, e_Q \rangle p$  such that  $\phi|_{pMp} = \tau_{pMp}$ . Recall that if  $S$  is a subset of a von Neumann algebra  $\mathcal{M}$ , then a state  $\phi$  on  $\mathcal{M}$  is said to be  *$S$ -central* if  $\phi(xT) = \phi(Tx)$ , for all  $x \in S$  and  $T \in \mathcal{M}$ .

**Remark 2.2.** Let  $P \subset pMp$  and  $Q \subset M$  be von Neumann subalgebras.

- (1) Suppose that there exists a non-zero projection  $p_0 \in P$  such that  $p_0 P p_0$  is amenable relative to  $Q$  inside  $M$ . Let  $p_1 \in \mathcal{Z}(P)$  be the central support of  $p_0$ . Then  $P p_1$  is amenable relative to  $Q$ . Indeed, let  $\xi_n \in L^2(p_0 \langle M, e_Q \rangle p_0)$  be a net such that  $\langle x \xi_n, \xi_n \rangle \rightarrow \tau(x)$ , for every  $x \in p_0 M p_0$ , and  $\|y \xi_n - \xi_n y\|_2 \rightarrow 0$ , for every  $y \in p_0 P p_0$ . Also, let  $\{v_i\}_{i=1}^\infty \subset P$  be partial isometries such that  $p_1 = \sum_{i=1}^\infty v_i v_i^*$  and  $v_i^* v_i \leq p_0$ , for all  $i$ . It is easy to see that the net  $\eta_n = \sum_{i=1}^\infty v_i \xi_n v_i^* \in L^2(p_1 \langle M, e_Q \rangle p_1)$  witnesses the fact that  $P p_1$  is amenable relative to  $Q$ .
- (2) Suppose that there exists a non-zero projection  $p_1 \in P' \cap pMp$  such that  $P p_1$  is amenable relative to  $Q$  inside  $M$ . Let  $p_2 \in \mathcal{Z}(P' \cap pMp)$  be the central support of  $p_1$ . By reasoning as in part (1) one deduces that  $P p_2$  is amenable relative to  $Q$  inside  $M$ .
- (3) If  $P \prec_M Q$ , then there is a non-zero projection  $p_0 \in P$  such that  $p_0 P p_0$  is amenable relative to  $Q$ . Thus by (1) and (2) there is a non-zero projection  $p_2 \in \mathcal{Z}(P' \cap pMp)$  such that  $P p_2$  is amenable relative to  $Q$  inside  $M$ .

The following lemma, established in [OP07, Corollary 2.3] (see also [PV11, Section 2.5]), provides a very useful criterion for relative amenability.

**Lemma 2.3.** [OP07] *Let  $(M, \tau)$  be a tracial von Neumann algebra and  $Q \subset M$  be a von Neumann subalgebra. Let  $P \subset pMp$  be a von Neumann subalgebra, for some projection  $p \in M$ . Assume that there exists a  $Q$ - $M$  bimodule  $\mathcal{K}$  and a net  $\xi_n \in pL^2(M) \otimes_Q \mathcal{K}$  such that*

- $\limsup_n \|x \xi_n\|_2 \leq \|x\|_2$ , for all  $x \in pMp$ ,
- $\limsup_n \|\xi_n\|_2 > 0$ , and
- $\|y \xi_n - \xi_n y\|_2 \rightarrow 0$ , for all  $y \in P$ .

*Then  $P p'$  is amenable relative to  $Q$  inside  $M$ , for some non-zero projection  $p' \in \mathcal{Z}(P' \cap pMp)$ .*

*Proof.* Let us first argue that we may additionally assume that  $\liminf_n \|\xi_n\|_2 > 0$ . To see this, suppose that the net  $\xi_n$  is indexed by a directed set  $I$  and denote  $\delta = \limsup_n \|\xi_n\|_2$ . Let  $J$  be set of triples  $j = (X, Y, \varepsilon)$ , where  $X \subset pMp, Y \subset P$  are finite sets and  $\varepsilon > 0$ . We make  $J$  a directed set by putting  $(X, Y, \varepsilon) \leq (X', Y', \varepsilon')$  if  $X \subset X', Y \subset Y'$  and  $\varepsilon' \leq \varepsilon$ .

Fix  $j = (X, Y, \varepsilon) \in J$ . By the hypothesis we can find  $n \in I$  such that  $\|x \xi_m\|_2 \leq \|x\|_2 + \varepsilon$  and  $\|y \xi_m - \xi_m y\|_2 \leq \varepsilon$ , for all  $x \in X, y \in Y$  and every  $m \geq n$ . Since  $\sup_{m \geq n} \|\xi_m\|_2 \geq \limsup_n \|\xi_n\|_2$ , we can find  $m \geq n$  such that  $\|\xi_m\|_2 > \frac{\delta}{2}$ . Define  $\eta_j = \xi_m$ . Then the net  $(\eta_j)_{j \in J}$  clearly satisfies  $\limsup_j \|x \eta_j\|_2 \leq \|x\|_2$ , for all  $x \in pMp$ ,  $\liminf_j \|\eta_j\|_2 > 0$ , and  $\|y \eta_j - \eta_j y\|_2 \rightarrow 0$ , for all  $y \in P$ .

Now, choose a state, denoted  $\lim_j$ , on  $\ell^\infty(J)$  extending the usual limit. Note that  $\pi : \langle M, e_Q \rangle \rightarrow \mathbb{B}(L^2(M) \otimes_Q \mathcal{K})$  given by  $\pi(T)(\xi \otimes_Q \eta) = T(\xi) \otimes_Q \eta$  is a normal  $*$ -homomorphism. Define  $\psi : \langle M, e_Q \rangle \rightarrow \mathbb{C}$  by letting

$$\psi(T) = \lim_j \|\eta_j\|_2^{-2} \langle \pi(T) \eta_j, \eta_j \rangle.$$

Then  $\psi$  is a state on  $\langle M, e_Q \rangle$  such that  $\psi(p) = 1$ ,  $\psi$  is  $P$ -central and  $\psi|_{pMp}$  is normal. By choosing, as in the proof of [OP07, Corollary 2.3], the minimal projection  $p' \in \mathcal{Z}(P' \cap pMp)$  such that  $\psi(p') = 1$  and applying [OP07, Theorem 2.1], the conclusion follows.  $\square$

**Lemma 2.4.** *Let  $(M, \tau)$  be a tracial von Neumann algebra and  $Q \subset M$  be a von Neumann subalgebra. Let  $P \subset pMp$  be a von Neumann subalgebra, for some projection  $p \in M$ . Let  $\omega$  be a free ultrafilter on  $\mathbb{N}$ .*

*Suppose that  $P \prec_{M^\omega} Q^\omega$ . More generally, assume that there exists a non-zero projection  $p_0 \in P' \cap (pMp)^\omega$  such that  $P p_0$  is amenable relative to  $Q^\omega$  inside  $M^\omega$ .*



Then  $Pp'$  is amenable relative to  $Q$  inside  $M$ , for some non-zero projection  $p' \in \mathcal{Z}(P' \cap pMp)$ .

*Proof.* Let  $X \subset pMp$ ,  $Y \subset P$  be finite subsets and  $\varepsilon > 0$ . Since  $Pp_0$  is amenable relative to  $Q^\omega$ , we can find a vector  $\xi \in L^2(p_0\langle M^\omega, e_{Q^\omega} \rangle p_0)$  such that

$$(2.1) \quad \|x\xi\|_2 < \|x\|_2 \quad \text{for all } x \in X, \quad \|\xi\|_2 > \frac{\|p_0\|_2}{2}, \quad \text{and}$$

$$(2.2) \quad \|y\xi - \xi y\|_2 < \varepsilon \quad \text{for all } y \in Y.$$

By approximating  $\xi$  in  $\|\cdot\|_2$ , we may assume that  $\xi$  is in the linear span of  $\{ae_{Q^\omega}b \mid a, b \in M^\omega\}$ . Write  $\xi = \sum_{i=1}^k a_i e_{Q^\omega} b_i$ , where  $a_i, b_i \in M^\omega$ . For every  $i \in \{1, \dots, k\}$ , represent  $a_i = (a_{i,n})_n$  and  $b_i = (b_{i,n})_n$ , where  $a_{i,n}, b_{i,n} \in M$ . For every  $n$ , define  $\xi_n = \sum_{i=1}^k a_{i,n} e_Q b_{i,n} \in \langle M, e_Q \rangle$ .

Then for all  $z \in M$ , we have that  $\|z\xi\|_2 = \lim_{n \rightarrow \omega} \|z\xi_n\|_2$  and  $\|\xi z\|_2 = \lim_{n \rightarrow \omega} \|\xi_n z\|_2$ . Using 2.1 and 2.2 it follows that we can find  $n$  such that  $\eta = \xi_n \in \langle M, e_Q \rangle$  satisfies  $\|x\eta\|_2 < \|x\|_2$ , for all  $x \in X$ ,  $\|\eta\|_2 > \frac{\|p\|_2}{2}$ , and  $\|y\xi - \xi y\|_2 < \varepsilon$ , for all  $y \in Y$ . Continuing as in the proof of Lemma 2.3 gives the conclusion.  $\square$

**2.3. Property  $\Gamma$ .** A  $\text{II}_1$  factor  $M$  has *property  $\Gamma$*  of Murray and von Neumann [MvN43] if there exists a sequence of unitaries  $u_n \in M$  with  $\tau(u_n) = 0$  such that  $\|xu_n - u_n x\|_2 \rightarrow 0$ , for all  $x \in M$ . If  $\omega$  is a free ultrafilter on  $\mathbb{N}$ , then property  $\Gamma$  is equivalent to  $M' \cap M^\omega \neq \mathbb{C}1$ .

By a well-known result of A. Connes [Co76, Theorem 2.1] property  $\Gamma$  is also equivalent to the existence of a net of unit vectors  $\xi_n \in L^2(M) \ominus \mathbb{C}1$  such that  $\|x\xi_n - \xi_n x\|_2 \rightarrow 0$ , for all  $x \in M$ . The proof of [Co76, Theorem 2.1] moreover shows the following.

**Theorem 2.5.** [Co76] *Let  $P$  be a von Neumann subalgebra of a  $\text{II}_1$  factor  $M$  and  $\omega$  be a free ultrafilter on  $\mathbb{N}$ .*

*If  $P' \cap M^\omega = \mathbb{C}1$ , then there does not exist a net of unit vectors  $\xi_n \in L^2(M) \ominus \mathbb{C}1$  such that  $\|x\xi_n - \xi_n x\|_2 \rightarrow 0$ , for all  $x \in P$ .*

**Remark 2.6.** Let us very briefly explain why Theorem 2.5 follows by repeating verbatim part of the proof of [Co76, Theorem 2.1]. To this end, assume by contradiction that  $P' \cap M^\omega = \mathbb{C}1$  and that there is a net of unit vectors  $\xi_n \in L^2(M) \ominus \mathbb{C}1$  such that  $\|x\xi_n - \xi_n x\|_2 \rightarrow 0$ , for all  $x \in P$ .

Let  $S = \{u_1, u_2, \dots, u_k\}$  be a finite set of unitary operators in  $P$ . If  $S' \cap M^\omega$  is finite dimensional, then since  $P' \cap M = \mathbb{C}1$ , [Co76, Lemma 2.6] implies that after replacing  $S$  with a larger set of unitary operators in  $P$ , we may assume that  $S' \cap M^\omega = \mathbb{C}1$ . Thus, we can suppose that we are in one of the following two cases: (1)  $S' \cap M^\omega = \mathbb{C}1$ , or (2)  $S' \cap M^\omega$  is infinite dimensional. In either case, the proof of [Co76, Theorem 2.1], implication (c)  $\Rightarrow$  (b), provides a non-normal  $S$ -central state on  $M$ . Since  $P$  is a factor, [Co76, Lemma 2.5] implies that whenever  $p \in P$  is a non-zero projection and  $S \subset \mathcal{U}(pPp)$  is a finite set, there exists a non-normal  $S$ -central state on  $pMp$ .

The proof of [Co76, Theorem 2.1], implication (b)  $\Rightarrow$  (a), now shows that for every  $\delta > 0$ , we can find a projection  $e \in M$  such that  $\tau(e) = \frac{1}{2}$  and  $\|u_j e u_j^* - e\|_2 \leq \delta$ , for all  $j \in \{1, 2, \dots, k\}$ . Since  $S \subset \mathcal{U}(P)$  is an arbitrary finite set, this implies that  $P' \cap M^\omega \neq \mathbb{C}1$ , providing the desired contradiction.

Next, we prove that the maximal central projection  $e$  of  $P' \cap M^\omega$  such that  $(P' \cap M^\omega)e$  is diffuse, belongs to  $M$ . More precisely, we have:

**Lemma 2.7.** *Let  $(M, \tau)$  be a tracial von Neumann algebra and  $P \subset pMp$  a von Neumann subalgebra, for a projection  $p \in M$ . Let  $\omega$  be a free ultrafilter on  $\mathbb{N}$  and denote  $P_\omega = P' \cap (pMp)^\omega$ .*

*Then we can find a projection  $e \in \mathcal{Z}(P' \cap pMp) \cap \mathcal{Z}(P_\omega)$  such that*

- (1)  $P_\omega e$  is completely atomic and  $P_\omega e = (P' \cap pMp)e$ .
- (2)  $P_\omega(p - e)$  is diffuse.

*Proof.* Let  $e \in \mathcal{Z}(P_\omega)$  be the maximal projection such that  $P_\omega e$  is completely atomic.

Let us prove that  $e \in \mathcal{Z}(P' \cap pMp)$ . To this end, write  $e = (e_n)_n$ , where  $e_n \in pMp$  is a projection, and let  $a$  be the weak limit of  $e_n$ , as  $n \rightarrow \omega$ . We have the following:

**Claim.** Let  $f_1, f_2, \dots, f_m \in M^\omega$ . Then we can find a subsequence  $\{k_n\}_{n \geq 1}$  of  $\mathbb{N}$  such that the projection  $f = (e_{k_n})_n \in (pMp)^\omega$  satisfies  $f \in P_\omega$  and

$$\tau_\omega(e f) = \tau(a^2), \quad \tau_\omega(e f a) = \tau(a^3) \quad \text{and} \quad \tau_\omega(e f_j f) = \tau_\omega(e f_j a), \quad \text{for all } j \in \{1, 2, \dots, m\}.$$

*Proof of the claim.* Let  $\{x_i\}_{i \geq 1}$  be a  $\|\cdot\|_2$  dense sequence of  $(P)_1$  and write  $f_j = (f_{j,n})_n$ , for  $j \in \{1, 2, \dots, m\}$ . Recall that  $\|x_i e_n - e_n x_i\|_2 \rightarrow 0$ , for all  $i$ , and that  $e_n \rightarrow a$ , weakly, as  $n \rightarrow \omega$ . Therefore, for every  $n \geq 1$  we can find  $k_n \geq 1$  such that

$$\|x_i e_{k_n} - e_{k_n} x_i\|_2 \leq \frac{1}{n}, \quad \text{for all } i \in \{1, 2, \dots, n\}, \quad |\tau(e_n e_{k_n}) - \tau(e_n a)| \leq \frac{1}{n},$$

$$|\tau(e_n e_{k_n} a) - \tau(e_n a^2)| \leq \frac{1}{n} \quad \text{and} \quad |\tau(e_n f_{j,n} e_{k_n}) - \tau(e_n f_{j,n} a)| \leq \frac{1}{n}, \quad \text{for all } j \in \{1, 2, \dots, m\}.$$

This inequalities clearly imply that  $f = (e_{k_n})_n$  satisfies the claim.  $\square$

Now, using the claim we can inductively construct a sequence of projections  $\{f_m\}_{m \geq 1} \in P_\omega$  such that  $\tau_\omega(e f_m) = \tau(a^2)$ ,  $\tau_\omega(e f_m a) = \tau(a^3)$  and  $\tau_\omega(e f_j f_m) = \tau_\omega(e f_j a)$ , for all  $j \in \{1, 2, \dots, m-1\}$  and  $m \geq 1$ . But then it follows that  $\tau(e f_j f_m) = \tau(a^3)$ , for all  $1 \leq j < m$ .

Next, for  $m \geq 1$ , let  $p_m = e f_m$ . Since  $e$  belongs to the center of  $P_\omega$ , we deduce that  $\{p_m\}_{m \geq 1} \in P_\omega e$  are projections such that  $\tau_\omega(p_m) = \tau(a^2)$  and  $\tau_\omega(p_j p_m) = \tau(a^3)$ , for all  $1 \leq j < m$ .

Finally, since  $P_\omega e$  is completely atomic, its unit ball is compact in  $\|\cdot\|_2$ . Thus we can find a subsequence  $\{p_{m_l}\}_{l \geq 1}$  of  $\{p_m\}_{m \geq 1}$  which is convergent in  $\|\cdot\|_2$ . In particular, we have that  $|\tau_\omega(p_{m_l} p_{m_k}) - \tau_\omega(p_{m_l})| \leq \|p_{m_l} - p_{m_k}\|_{2,\omega} \rightarrow 0$ , as  $l, k \rightarrow \infty$ . This implies that  $\tau(a^2) = \tau(a^3)$ . Since  $0 \leq a \leq 1$ ,  $a$  must be a projection. Thus we have that  $\|e_n - a\|_2^2 = \tau(e_n) + \tau(a) - 2\tau(e_n a) \rightarrow 0$ , as  $n \rightarrow \omega$ . Hence  $e = (e_n)_n = a \in pMp$  and so  $e \in P' \cap pMp$ . Since  $P'_\omega \cap pMp \subset (P' \cap pMp)' \cap pMp$ , it follows that  $e \in \mathcal{Z}(P' \cap pMp)$ .

Let  $P_0 = P e$ . Since  $e \in M$ , we have that  $P_0$  is a subalgebra of  $e M e$  and  $P'_0 \cap (e M e)^\omega = P_\omega e$  is completely atomic. The proof of [Co76, Lemma 2.6] then gives that  $P'_0 \cap (e M e)^\omega \subset e M e$ . Thus  $P_\omega e \subset e M e$  and hence  $P_\omega e = (P' \cap pMp)e$ . This proves that  $e$  satisfies the first assertion. The second assertion is immediate by the maximality of  $e$ .  $\square$

**2.4. Normalizers in crossed products by free groups.** Very recently, S. Popa and S. Vaes have established the following remarkable dichotomy.

**Theorem 2.8.** [PV11] *Let  $\mathbb{F}_n \curvearrowright (N, \tau)$  be a trace preserving action of a free group on a tracial von Neumann algebra  $(N, \tau)$ . Denote  $M = N \rtimes \mathbb{F}_n$  and let  $A \subset pMp$  be a von Neumann subalgebra that is amenable relative to  $N$ , for some projection  $p \in M$ .*

*Then either  $A \prec_M N$  or  $\mathcal{N}_M(A)''$  is amenable relative to  $N$  inside  $M$ .*

More generally, it is proven in [PV11, Theorem 1.6] that the same holds when  $\mathbb{F}_n$  is replaced by a group  $\Gamma$  that admits a proper cocycle into an orthogonal representation that is weakly contained in the regular representation.

**2.5. Deformations of AFP algebras.** Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be two tracial von Neumann algebras with a common von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B$ . Denote by  $M = M_1 *_B M_2$  the amalgamated free product algebra (abbreviated, **AFP algebra**) and by  $\tau$  its trace extending  $\tau_1$  and  $\tau_2$ . To present the canonical decomposition of  $L^2(M)$ , let us fix some notations:

**Notations 2.9.** Let  $n \geq 1$

- We denote by  $S_n = \{(1, 2, 1, \dots), (2, 1, 2, \dots)\}$  the set consisting of the two alternating sequences of 1's and 2's of length  $n$ .
- For  $\mathcal{I} = (i_1, i_2, \dots, i_n) \in S_n$ , we denote  $\mathcal{H}_{\mathcal{I}} = L^2(M_{i_1} \ominus B) \otimes_B \dots \otimes_B L^2(M_{i_n} \ominus B)$ .
- We also let  $\mathcal{H}_n = \bigoplus_{\mathcal{I} \in S_n} \mathcal{H}_{\mathcal{I}}$  and  $\mathcal{H}_0 = L^2(B)$ .

With these notations, we have  $L^2(M) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ . This decomposition easily implies the following lemma that will be useful in the sequel:

**Lemma 2.10.** *Let  $(M_1, \tau_1)$ ,  $(M_2, \tau_2)$ ,  $(M_3, \tau_3)$  be tracial von Neumann algebras with a common von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B = \tau_3|_B$ . Then*

- (1) *We can find a  $B$ - $M_1$  bimodule  $\mathcal{H}$  and a  $M_1$ - $B$  bimodule  $\mathcal{K}$  such that, as  $M_1$ - $M_1$  bimodules, we have  $L^2(M_1 *_B M_2) \ominus L^2(M_1) \cong L^2(M_1) \otimes_B \mathcal{H} \cong \mathcal{K} \otimes_B L^2(M_1)$ .*
- (2) *We can find a  $B$ - $B$  bimodule  $\mathcal{L}$  such that  $L^2(M_1 *_B M_2 *_B M_3) \cong L^2(M_1) \otimes_B \mathcal{L} \otimes_B L^2(M_2)$ , as  $M_1$ - $M_2$  bimodules.*

Let us recall from [IPP05, Section 2.2] the construction of the *free malleable deformation* of  $M = M_1 *_B M_2$ . Define  $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{F}_2))$ . Denote  $u_1 = u_{a_1}$ ,  $u_2 = u_{a_2}$ , where  $a_1, a_2$  are generators of  $\mathbb{F}_2$ . Note that we can decompose  $\tilde{M} = \tilde{M}_1 *_B \tilde{M}_2$ , where  $\tilde{M}_1 = M_1 *_B (B \bar{\otimes} L(\mathbb{Z}))$  and  $\tilde{M}_2 = M_2 *_B (B \bar{\otimes} L(\mathbb{Z}))$ , and the two copies of  $\mathbb{Z}$  are the cyclic groups generated by  $a_1$  and  $a_2$ , respectively.

Consider the unique function  $f : \mathbb{T} \rightarrow (-\pi, \pi]$  satisfying  $f(\exp(it)) = t$ , for all  $t \in (-\pi, \pi]$ . Then  $\alpha_1 = f(u_1)$  and  $\alpha_2 = f(u_2)$  are hermitian operators such that  $u_1 = \exp(i\alpha_1)$  and  $u_2 = \exp(i\alpha_2)$ . For  $t \in \mathbb{R}$ , define the unitary elements  $u_1^t = \exp(it\alpha_1)$  and  $u_2^t = \exp(it\alpha_2)$ .

Since the restrictions of the automorphisms  $\text{Ad}(u_1^t)$  and  $\text{Ad}(u_2^t)$  of  $\tilde{M}_1$  and  $\tilde{M}_2$  to  $B$  are equal (to  $\text{id}_B$ ), the formulae

$$\theta_t(x) = u_1^t x u_1^{t*}, \text{ for } x \in \tilde{M}_1, \text{ and } \theta_t(y) = u_2^t y u_2^{t*}, \text{ for } y \in \tilde{M}_2,$$

define a 1-parameter group  $\{\theta_t\}_{t \in \mathbb{R}}$  automorphisms of  $\tilde{M}$ .

The following is the main technical result of [IPP05].

**Theorem 2.11.** [IPP05] *Let  $A \subset pMp$  be a von Neumann subalgebra, for a projection  $p \in M$ . Assume that there exist  $c > 0$  and  $t > 0$  such that  $\tau(\theta_t(u)u^*) \geq c$ , for all  $u \in \mathcal{U}(A)$ .*

*Then either  $A \prec_M B$ , or  $\mathcal{N}_M(A)'' \prec_M M_i$ , for some  $i \in \{1, 2\}$ .*

Theorem 2.11 is formulated in a different way and proved under an additional assumption in [IPP05, Theorem 3.1]. For the formulation given here, see [Ho07, Section 5] and [PV09, Theorem 5.4].

Note that since  $\tau(u_1^t) = \tau(u_2^t) = \frac{\sin(\pi t)}{\pi t}$ , we have that  $E_M(\theta_t(x)) = (\frac{\sin(\pi t)}{\pi t})^{2n} x$ , for all  $x \in \mathcal{H}_n$ . Thus, if we write  $x \in M$  as  $x = \sum_{n \geq 0} x_n$ , where  $x_n \in \mathcal{H}_n$ , then we have

$$(2.3) \quad \tau(\theta_t(x)x^*) = \tau(E_M(\theta_t(x))x^*) = \sum_{n \geq 0} \left(\frac{\sin(\pi t)}{\pi t}\right)^{2n} \|x_n\|_2^2.$$

We derive next a consequence of Theorem 2.11 that we will need in the proof of Theorem 6.3.

**Corollary 2.12.** *Let  $A \subset pMp$  be a von Neumann subalgebra, for some projection  $p \in M$ .*

*If  $A$  is amenable relative to  $M_1$ , then either  $A$  is amenable relative to  $B$  or  $\mathcal{N}_{pMp}(A)'' \prec_M M_1$ .*

*Proof.* Assume that  $A$  is amenable relative to  $M_1$ . In the first part of the proof we show that either  $Ap'$  is amenable relative to  $B$ , for a non-zero projection  $p' \in \mathcal{Z}(A' \cap pMp)$ , or  $\mathcal{N}_{pMp}(A)'' \prec_M M_1$ . To do this, we follow closely the strategy of proof of [OP07, Theorem 4.9].

Since  $A$  is amenable relative to  $M_1$  we can find a net  $\{\xi_n\}_{n \in I} \in L^2(p\langle M, e_{M_1} \rangle p)$  such that

$$(2.4) \quad \|x\xi_n - \xi_n x\|_2 \rightarrow 0, \text{ for all } x \in A, \text{ and}$$

$$(2.5) \quad \langle y\xi_n, \xi_n \rangle \rightarrow \tau(y), \text{ for all } y \in pMp.$$

Moreover, the proof of [OP07, Theorem 2.1] shows that  $\xi_n$  can be chosen such that  $\xi_n = \zeta_n^{\frac{1}{2}}$ , for some  $\zeta_n \in L^1(\langle M, e_{M_1} \rangle)_+$ . Thus,  $\langle \xi_n y, \xi_n \rangle = Tr(\zeta_n y) = \langle y\xi_n, \xi_n \rangle \rightarrow \tau(y)$ , for all  $y \in pMp$ .

Next, for  $t \in \mathbb{R}$ , we consider the automorphism  $\alpha_t$  of  $\tilde{M}$  given by  $\alpha_t(x) = x$ , for all  $x \in \tilde{M}_1$ , and  $\alpha_t(y) = u_2^t y u_2^{t*}$ , for all  $y \in \tilde{M}_2$ . Since  $\alpha_t$  is an automorphism of  $\tilde{M}$  that leaves  $M_1$  invariant we can extend it to an automorphism of  $\langle \tilde{M}, e_{M_1} \rangle$  by letting  $\alpha_t(e_{M_1}) = e_{M_1}$ .

We also let  $\mathcal{H}$  be the  $\|\cdot\|_2$  closure of the span of  $Me_{M_1}\tilde{M} = \{xe_{M_1}y | x \in M, y \in \tilde{M}\}$  and denote by  $e$  the orthogonal projection from  $L^2(\langle \tilde{M}, e_{M_1} \rangle)$  onto  $\mathcal{H}$ .

**Claim.** Let  $x \in A, y \in \tilde{M}$  and  $t \in \mathbb{R}$ . Then we have

- (1)  $\lim_n \|y\alpha_t(\xi_n)\|_2^2 = \tau(y^*y\alpha_t(p)) \leq \|y\|_2^2$  and  $\lim_n \|\alpha_t(\xi_n)y\|_2^2 = \tau(yy^*\alpha_t(p)) \leq \|y\|_2^2$ .
- (2)  $\limsup_n \|ye(\alpha_t(\xi_n))\|_2 \leq \|y\|_2$ .
- (3)  $\limsup_n \|x\alpha_t(\xi_n) - \alpha_t(\xi_n)x\|_2 \leq 2\|\alpha_t(x) - x\|_2$ .

*Proof of the claim.* (1) Since  $\xi_n \in p\mathcal{H}$  and  $(\tilde{M} \ominus M)\mathcal{H} \perp \mathcal{H}$ , by using 2.5 we get that

$$\begin{aligned} \|y\alpha_t(\xi_n)\|_2^2 &= \langle \alpha_t^{-1}(y^*y)\xi_n, \xi_n \rangle = \langle E_M(\alpha_t^{-1}(y^*y))\xi_n, \xi_n \rangle = \\ &= \langle pE_M(\alpha_t^{-1}(y^*y))p\xi_n, \xi_n \rangle \longrightarrow \tau(pE_M(\alpha_t^{-1}(y^*y))p) = \tau(y^*y\alpha_t(p)). \end{aligned}$$

The second inequality follows similarly using the fact that  $\langle \xi_n y, \xi_n \rangle \rightarrow \tau(y)$ , for all  $y \in pMp$ .

(2) Since  $(\tilde{M} \ominus M)\mathcal{H} \perp \mathcal{H}$  and  $\mathcal{H}$  is a left  $M$ -module, we derive that

$$\begin{aligned} \|ye(\alpha_t(\xi_n))\|_2^2 &= \langle y^*ye(\alpha_t(\xi_n)), e(\alpha_t(\xi_n)) \rangle = \langle E_M(y^*y)e(\alpha_t(\xi_n)), e(\alpha_t(\xi_n)) \rangle = \\ &= \|e(E_M(y^*y)^{\frac{1}{2}}\alpha_t(\xi_n))\|_2^2 \leq \|E_M(y^*y)^{\frac{1}{2}}\alpha_t(\xi_n)\|_2^2. \end{aligned}$$

On the other hand, by (1) we have that  $\|E_M(y^*y)^{\frac{1}{2}}\alpha_t(\xi_n)\|_2 \leq \|E_M(y^*y)^{\frac{1}{2}}\|_2 = \|y\|_2$ .

(3) Since  $\|x\alpha_t(\xi_n) - \alpha_t(\xi_n)x\|_2 \leq \|(x - \alpha_t(x))\alpha_t(\xi_n)\|_2 + \|\alpha_t(\xi_n)(x - \alpha_t(x))\|_2 + \|x\xi_n - \xi_n x\|_2$ , the inequality follows by combining (1) and 2.4.  $\square$

Let  $J = (0, \infty) \times I$ . Given  $(t, n) \in J$ , we denote  $\eta_{t,n} = \alpha_t(\xi_n) - e(\alpha_t(\xi_n))$  and  $\delta_{t,n} = \|\eta_{t,n}\|_2$ . For the rest of the proof we treat two separate cases.

**Case 1.** We can find  $t > 0$  such that  $\limsup_n \delta_{t,n} < \frac{\|p\|_2}{2}$ .

**Case 2.** For all  $t > 0$  we have that  $\limsup_n \delta_{t,n} \geq \frac{\|p\|_2}{2}$ .

In **Case 1**, fix  $x \in \mathcal{U}(A)$ . Since  $\mathcal{H}$  is a left  $M$ -module and  $(\tilde{M} \ominus M)\mathcal{H} \perp \mathcal{H}$  we get that

$$(2.6) \quad \begin{aligned} \|E_M(\alpha_t(x))\alpha_t(\xi_n)\|_2 &\geq \|e(E_M(\alpha_t(x))\alpha_t(\xi_n))\|_2 = \|e(\alpha_t(x)e(\alpha_t(\xi_n)))\|_2 \geq \\ &\geq \|e(\alpha_t(x)\alpha_t(\xi_n))\|_2 - \delta_{t,n} \geq \|e(\alpha_t(\xi_n)\alpha_t(x))\|_2 - \|x\xi_n - \xi_n x\|_2 - \delta_{t,n} \end{aligned}$$

On the other hand, since  $\mathcal{H}$  is a right  $\tilde{M}$ -module we deduce that

$$(2.7) \quad \|e(\alpha_t(\xi_n)\alpha_t(x))\|_2 = \|e(\alpha_t(\xi_n))\alpha_t(x)\|_2 \geq \|\alpha_t(\xi_n)\alpha_t(x)\|_2 - \delta_{t,n} = \|\xi_n x\|_2 - \delta_{t,n}$$

By combining part (1) of the Claim with equations 2.6, 2.7, 2.4 and 2.5 we derive that

$$(2.8) \quad \begin{aligned} \|E_M(\alpha_t(x))\|_2 &\geq \lim_n \|E_M(\alpha_t(x))\alpha_t(\xi_n)\|_2 \geq \\ &\liminf_n (\|\xi_n x\|_2 - \|x\xi_n - \xi_n x\|_2 - 2\delta_{t,n}) = \\ &\|x\|_2 - 2 \limsup_n \delta_{t,n} = \|p\|_2 - 2 \limsup_n \delta_{t,n} > 0, \quad \text{for all } x \in \mathcal{U}(A). \end{aligned}$$

Now, recall from notations 2.9 that  $L^2(M) = \mathcal{H}_0 \oplus_{m \geq 1} (\oplus_{\mathcal{I} \in S_m} \mathcal{H}_{\mathcal{I}})$ . Thus, we can write  $x = x_0 + \sum_{\mathcal{I} \in S_m} x_{\mathcal{I}}$ , where  $x_{\mathcal{I}} \in \mathcal{H}_{\mathcal{I}}$ . It is easy to see that if  $c_{\mathcal{I}}$  denotes the number of times 2 appears in  $\mathcal{I}$ , then  $E_M(\alpha_t(x_{\mathcal{I}})) = (\frac{\sin(\pi t)}{\pi t})^{2c_{\mathcal{I}}} x_{\mathcal{I}}$ . Therefore,  $\|E_M(\alpha_t(x))\|_2^2 = \|x_0\|_2^2 + \sum_{\mathcal{I} \in S_m} (\frac{\sin(\pi t)}{\pi t})^{4c_{\mathcal{I}}} \|x_{\mathcal{I}}\|_2^2$ .

On the other hand, by 2.3 we have  $\tau(\theta_t(x)x^*) = \|x_0\|_2^2 + \sum_{\mathcal{I} \in S_m} (\frac{\sin(\pi t)}{\pi t})^{2m} \|x_{\mathcal{I}}\|_2^2$ . Since every  $\mathcal{I} \in S_m$  is an alternating sequence of 1's and 2's, we have that  $2c_{\mathcal{I}} \geq m - 1$ .

By combining the last three facts, we conclude that  $\tau(\theta_t(x)x^*) \geq (\frac{\sin(\pi t)}{\pi t})^2 \|E_M(\alpha_t(x))\|_2^2$ , for every  $x \in M$ . Together with 2.8 this implies that  $\inf_{x \in \mathcal{U}(A)} \tau(\theta_t(x)x^*) > 0$ .

Thus, by Theorem 2.11 we get that either  $A \prec_M M_1$  or  $A \prec_M M_2$ . If  $A \prec_M M_1$ , then [IPP05, Theorem 1.1] gives that either  $A \prec_M B$  or  $\mathcal{N}_M(A)'' \prec_M M_1$ . Since by Remark 2.2, having  $A \prec_M B$  implies that there exists a non-zero projection  $p' \in \mathcal{Z}(A' \cap pMp)$  such that  $Ap'$  is amenable relative to  $B$ , the conclusion follows in this case.

Therefore, in order to finish the proof of **Case 1** we only need to analyze the case when  $A \prec_M M_2$ . By Remark 2.2 we can find a non-zero projection  $p' \in \mathcal{Z}(A' \cap pMp)$  such that  $Ap'$  is amenable relative to  $M_2$ . By the hypothesis we have that  $A$  and thus  $Ap'$  is amenable relative to  $M_1$ .

We claim that  $Ap'$  is amenable relative to  $B$ . To this end, denote  $\mathcal{K} = L^2(\langle M, e_{M_1} \rangle) \otimes_M L^2(\langle M, e_{M_2} \rangle)$ . Lemma 2.10 provides a  $B$ - $B$  bimodule  $\mathcal{L}$  such that  $L^2(M) \cong L^2(M_1) \otimes_B \mathcal{L} \otimes_B L^2(M_2)$ , as  $M_1$ - $M_2$  bimodules. Thus, we have the following isomorphisms of  $M$ - $M$  bimodules

$$\begin{aligned} \mathcal{K} &\cong (L^2(M) \otimes_{M_1} L^2(M)) \otimes_M (L^2(M) \otimes_{M_2} L^2(M)) \cong L^2(M) \otimes_{M_1} L^2(M) \otimes_{M_2} L^2(M) \cong \\ &L^2(M) \otimes_{M_1} (L^2(M_1) \otimes_B \mathcal{L} \otimes_B L^2(M_2)) \otimes_{M_2} L^2(M) \cong \\ &L^2(M) \otimes_B \mathcal{L} \otimes_B L^2(M). \end{aligned}$$

Since  $Ap'$  is amenable relative to both  $M_1$  and  $M_2$ , the first part of the proof of [PV11, Proposition 2.7] implies that the  $p'Mp'$ - $Ap'$  bimodule  $L^2(p'Mp')$  is weakly contained in the  $p'Mp'$ - $Ap'$  bimodule  $p'\mathcal{K}p'$ . Thus the  $p'Mp'$ - $Ap'$  bimodule  $p'L^2(M) \otimes_B \mathcal{L} \otimes_B L^2(M)p'$  weakly contains the  $p'Mp'$ - $Ap'$  bimodule  $L^2(p'Mp')$ . By Lemma 2.3 it follows that  $Ap'$  is amenable relative to  $B$ . This completes the proof of **Case 1**.

In **Case 2**, we claim that there exists a net  $(\eta_k)$  in  $\mathcal{H}$  such that  $\|x\eta_k - \eta_k x\|_2 \rightarrow 0$ , for all  $x \in A$ ,  $\limsup_k \|y\eta_k\|_2 \leq 2\|y\|_2$ , for all  $y \in pMp$ , and  $\limsup_k \|p\eta_k\|_2 > 0$ .

Towards this, let  $k = (X, Y, \varepsilon)$  be a triple such that  $X \subset A$ ,  $Y \subset pMp$  are finite sets and  $\varepsilon > 0$ . Then we can find  $t > 0$  such that

$$(2.9) \quad \|\alpha_t(x) - x\|_2 < \frac{\varepsilon}{2}, \quad \text{for all } x \in X, \quad \text{and} \quad \|\alpha_t(p) - p\|_2 < \frac{\|p\|_2}{10}.$$

Let  $x \in X$  and  $y \in Y$ . Firstly, since  $\eta_{t,n} = (1-e)(\alpha_t(\xi_n))$  and  $x \in M$  we get that  $\|x\eta_{t,n} - \eta_{t,n}x\|_2 \leq \|x\alpha_t(\xi_n) - \alpha_t(\xi_n)x\|_2$ . This inequality together with part (3) of the Claim and 2.9 implies that  $\limsup_n \|x\eta_{t,n} - \eta_{t,n}x\|_2 \leq 2\|\alpha_t(x) - x\|_2 < \varepsilon$ .

Secondly, by combining parts (1) and (2) of the Claim we get that  $\limsup_n \|y\eta_{t,n}\|_2 \leq 2\|y\|_2$ .

Thirdly, part (1) of the Claim gives that  $\limsup_n \|p\eta_{t,n}\|_2 \geq \limsup_n (\|p\alpha_t(\xi_n)\|_2 - \|e(\alpha_t(\xi_n))\|_2) = \|p\alpha_t(p)\|_2 - \liminf_n \|e(\alpha_t(\xi_n))\|_2$ . Also, since  $\|\xi_n\|_2 \rightarrow \|p\|_2$  we have that  $\liminf_n \|e(\alpha_t(\xi_n))\|_2 = \sqrt{\|p\|_2^2 - \limsup_n \|\eta_{t,n}\|_2^2} \leq \frac{\sqrt{3}}{2}\|p\|_2$ . Since 2.9 implies that  $\|p\alpha_t(p)\|_2 > \frac{9}{10}\|p\|_2$ , we altogether deduce that  $\limsup_n \|p\eta_{t,n}\|_2 > (\frac{9}{10} - \frac{\sqrt{3}}{2})\|p\|_2$ .

The last three paragraphs imply that for some  $n \in I$ ,  $\eta_k = \eta_{t,n}$  satisfies  $\|x\eta_k - \eta_kx\|_2 < \varepsilon$ , for all  $x \in X$ ,  $\|y\eta_k\|_2 \leq 2\|y\|_2 + \varepsilon$ , for all  $y \in Y$ , and  $\|p\eta_k\|_2 > (\frac{9}{10} - \frac{\sqrt{3}}{2})\|p\|_2$ . It is now clear that the net  $(\eta_k)$  has the desired properties.

Finally, by the definition of  $\mathcal{H}$ , the  $M$ - $M$  bimodule  $L^2(\langle \tilde{M}, e_{M_1} \rangle) \ominus \mathcal{H}$  is isomorphic to the  $M$ - $M$  bimodule  $(L^2(\tilde{M}) \ominus L^2(M)) \otimes_{M_1} L^2(\tilde{M})$ . Since  $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{F}_2))$ , Lemma 2.10 (1) provides a  $B$ - $M$  bimodule  $\mathcal{K}$  such that  $L^2(\tilde{M}) \ominus L^2(M) \cong L^2(M) \otimes_B \mathcal{K}$ . Thus, we have the following isomorphism of  $M$ - $M$  bimodules

$$L^2(\langle \tilde{M}, e_{M_1} \rangle) \ominus \mathcal{H} \cong L^2(M) \otimes_B (\mathcal{K} \otimes_{M_1} L^2(\tilde{M})).$$

Since  $\eta_k \in L^2(\langle \tilde{M}, e_{M_1} \rangle) \ominus \mathcal{H}$ , for all  $k$ , by Lemma 2.3 there is a non-zero projection  $p' \in \mathcal{Z}(A' \cap pMp)$  such that  $Ap'$  is amenable relative to  $B$ . This finishes the proof of **Case 2**.

Now, to get the conclusion, let  $p_0 \in \mathcal{Z}(A' \cap pMp)$  be the maximal projection such that  $Ap_0$  is amenable relative to  $B$ . It is easy to see that  $p_0 \in \mathcal{N}_{pMp}(A)' \cap pMp$ .

Let  $p_1 = p - p_0$ . If  $p_1 = 0$ , then  $A$  is amenable relative to  $B$ . If  $p_1 \neq 0$ , then  $Ap_1$  is amenable relative to  $M_1$ . By the first part of the proof either  $Ap'$  is amenable relative to  $B$ , for some non-zero projection  $p' \in \mathcal{Z}(A' \cap pMp)p_1$ , or  $\mathcal{N}_{p_1Mp_1}(Ap_1)'' \prec_M M_1$ . By the maximality of  $p_0$ , the former is impossible; since  $\mathcal{N}_{pMp}(A)p_1 \subset \mathcal{N}_{p_1Mp_1}(Ap_1)$ , the latter implies that  $\mathcal{N}_{pMp}(A)'' \prec_M M_1$ .  $\square$

**2.6. Random walks on countable groups.** We end this section with some facts from the theory of random walks on countable groups that we will need in Section 3. Let  $\mu$  and  $\nu$  be probability measures on a countable group  $\Gamma$ . The *support* of  $\mu$  is the set of  $g \in \Gamma$  with  $\mu(g) \neq 0$ . The convolution of  $\mu$  and  $\nu$  is the probability measure on  $\Gamma$  given by  $(\mu * \nu)(g) = \sum_{h \in \Gamma} \mu(gh^{-1})\nu(h)$ . For  $n \geq 1$ , we denote  $\mu^{*n} = \underbrace{\mu * \mu * \dots * \mu}_{n \text{ times}}$ .

The next lemma is well-known (see for instance [Fu02, Theorems 2.2 and 2.28]). For the reader's convenience, we include a proof.

**Lemma 2.13.** *Let  $\Gamma$  be a finitely generated group and denote by  $\ell_S : \Gamma \rightarrow \mathbb{N}$  the word length with respect to a finite set of generators  $S$ . Let  $\mu$  be a probability measure on  $\Gamma$  whose support generates a non-amenable subgroup and contains the identity element.*

- (1) *Then  $\mu^{*n}(g) \rightarrow 0$ , for all  $g \in \Gamma$ .*
- (2) *Assume that  $\sum_{g \in \Gamma} \ell_S(g)^p \mu(g) < +\infty$ , for some  $p \in (0, 1]$ . If  $\Sigma < \Gamma$  is a finitely generated nilpotent (e.g. cyclic) subgroup, then  $\mu^{*n}(h\Sigma k) \rightarrow 0$ , for all  $h, k \in \Gamma$ .*

*Proof.* (1) Let  $\lambda : \Gamma \rightarrow \mathcal{U}(\ell^2(\Gamma))$  be the left regular representation of  $\Gamma$ . Define the operator  $T : \ell^2(\Gamma) \rightarrow \ell^2(\Gamma)$  by  $T = \sum_{g \in \Gamma} \mu(g)\lambda(g)$ . Since the support of  $\mu$  generates a non-amenable group we have that  $\|T\| < \sum_{g \in \Gamma} \mu(g) = 1$ .

Denote by  $\{\delta_g\}_{g \in \Gamma}$  the canonical orthonormal basis of  $\ell^2(\Gamma)$ . Then for  $n \geq 1$  and  $g \in \Gamma$  we have

$$\mu^{*n}(g) = \sum_{\substack{g_1, g_2, \dots, g_n \in \Gamma \\ g_1 g_2 \dots g_n = g}} \mu(g_1) \mu(g_2) \dots \mu(g_n) = \langle T^n(\delta_e), \delta_g \rangle.$$

This implies that  $\mu^{*n}(g) \leq \|T\|^n$  and since  $\|T\| < 1$ , we are done.

(2) Define the product probability space  $(\Omega, \nu) = (\Gamma^{\mathbb{N}}, \mu^{\mathbb{N}})$  together with the shift  $T : \Omega \rightarrow \Omega$  given by  $(T\omega)_n = \omega_{n+1}$ , for all  $\omega = (\omega_n)_n \in \Omega$ . Then  $T$  is an ergodic, measure preserving transformation of  $(\Omega, \nu)$ . For  $n \geq 1$ , define  $X_n : \Omega \rightarrow \Gamma$  by letting  $X_n(\omega) = \omega_1 \omega_2 \dots \omega_n$ . Note that  $\mu^{*n} = (X_n)_*(\nu)$ .

Further, let  $p \in (0, 1]$  as in the hypothesis and define  $S_n : \Omega \rightarrow [0, \infty)$  by  $S_n(\omega) = \ell_S(X_n(\omega))^p$ . Since  $p \in (0, 1]$ , we have that  $(a+b)^p \leq a^p + b^p$ , for all  $a, b \geq 0$ . Recall that for every  $g, h \in \Gamma$  we have that  $\ell_S(gh) \leq \ell_S(g) + \ell_S(h)$ . Also we have that  $X_{n+m}(\omega) = X_n(\omega)X_m(T^n(\omega))$ , for all  $n, m \geq 1$  and  $\omega \in \Omega$ . By combining these three facts we deduce that

$$(2.10) \quad S_{n+m}(\omega) \leq S_n(\omega) + S_m(T^n(\omega)), \text{ for all } \omega \in \Omega \text{ and } n, m \geq 1$$

Additionally, by using the hypothesis we get that

$$(2.11) \quad \int_{\Omega} S_1(\omega) d\nu(\omega) = \int_{\Omega} \ell_S(X_1(\omega))^p d\nu(\omega) = \int_{\Gamma} \ell_S(\omega_1)^p d\mu(\omega_1) < +\infty$$

Since  $T$  is ergodic, equations 2.10 and 2.11 guarantee that we can apply Kingman's subadditive ergodic theorem. Thus, we can find a constant  $\alpha \in [0, \infty)$  such that  $\frac{1}{n}S_n(\omega) \rightarrow \alpha$ , for  $\nu$ -almost every  $\omega \in \Omega$ . It follows that  $\nu(\{\omega \in \Omega \mid S_n(\omega) > (\alpha + 1)n\}) \rightarrow 0$ , as  $n \rightarrow \infty$ .

Hence, if we let  $f(n) = ((\alpha + 1)n)^{\frac{1}{p}}$ , then  $\nu(\{\omega \in \Omega \mid \ell_S(X_n(\omega)) > f(n)\}) \rightarrow 0$ , as  $n \rightarrow \infty$ . Since  $(X_n)_*(\nu) = \mu^{*n}$ , we deduce that

$$(2.12) \quad \varepsilon_n := \mu^{*n}(\{g \in \Gamma \mid \ell_S(g) > f(n)\}) \rightarrow 0, \text{ as } n \rightarrow \infty$$

Now, since  $\Sigma$  is a finitely generated nilpotent group, it has polynomial growth. Thus, we can find  $a, b > 0$  such that  $|\{g \in \Sigma \mid \ell_S(g) \leq n\}| \leq an^b$ , for all  $n$ . Denoting  $c = \ell_S(h) + \ell_S(k)$ , we get that

$$(2.13) \quad |\{g \in h\Sigma k \mid \ell_S(g) \leq n\}| \leq a(n+c)^b, \text{ for all } n$$

Recall from the proof of part (1) that  $\mu^{*n}(g) \leq \|T\|^n$ , for all  $g \in \Gamma$  and  $n \geq 1$ . Combining this fact with 2.12 and 2.13 yields that

$$\begin{aligned} \mu^{*n}(h\Sigma k) &\leq \varepsilon_n + \mu^{*n}(\{g \in h\Sigma k \mid \ell_S(g) \leq f(n)\}) \leq \\ &\varepsilon_n + a\|T\|^n(f(n) + c)^b, \text{ for all } n \geq 1. \end{aligned}$$

As  $\varepsilon_n \rightarrow 0$ ,  $\|T\| < 1$  and  $f(n)$  grows polynomially in  $n$ , we conclude that  $\mu^{*n}(h\Sigma k) \rightarrow 0$ .  $\square$

### 3. A CONJUGACY RESULT FOR SUBALGEBRAS OF AFP ALGEBRAS

Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be two tracial von Neumann algebras with a common von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B$ . Denote  $M = M_1 *_B M_2$  and let  $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{F}_2))$ . For  $t \in \mathbb{R}$ , we consider the automorphism  $\theta_t : \tilde{M} \rightarrow \tilde{M}$  defined in Section 2.11. We denote by  $\{u_g\}_{g \in \mathbb{F}_2} \subset L(\mathbb{F}_2)$  the canonical unitaries and consider the notations from 2.9.

In this context, we have

**Lemma 3.1.** *Let  $\mathcal{I} = (i_1, i_2, \dots, i_n) \in S_n$  and  $\mathcal{J} = (j_1, j_2, \dots, j_m) \in S_m$ , for some  $n, m \geq 1$ . Let  $x_1 \in M_{i_1} \ominus B, x_2 \in M_{i_2} \ominus B, \dots, x_n \in M_{i_n} \ominus B$  and  $y_1 \in M_{j_1} \ominus B, y_2 \in M_{j_2} \ominus B, \dots, y_m \in M_{j_m} \ominus B$ . Let  $g_1, g_2, \dots, g_{n+1}, h_1, h_2, \dots, h_{m+1} \in \mathbb{F}_2$ .*

*Then*

$$\begin{aligned} & \langle u_{g_1} x_1 u_{g_2} x_2 \dots u_{g_n} x_n u_{g_{n+1}}, u_{h_1} y_1 u_{h_2} y_2 \dots u_{h_m} y_m u_{h_{m+1}} \rangle = \\ & \begin{cases} \langle x_1 x_2 \dots x_n, y_1 y_2 \dots y_m \rangle, & \text{if } n = m, \mathcal{I} = \mathcal{J}, \text{ and } g_k = h_k, \text{ for all } k \in \{1, 2, \dots, n+1\}, \text{ and} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

*Proof.* Denote  $A_0 = \{u_g\}_{g \in \mathbb{F}_2 \setminus \{e\}}$ ,  $A_1 = M_1 \ominus B$  and  $A_2 = M_2 \ominus B$ . We say that  $z = z_1 z_2 \dots z_n$  is an *alternating product* if for all  $i$  we have that  $z_i \in A_j$ , for some  $j \in \{0, 1, 2\}$  and that  $z_i$  and  $z_{i+1}$  belong to different  $A_j$ 's. It is clear that  $\tau(z) = 0$ , for any alternating product  $z$ .

We proceed by induction on  $\max\{n, m\}$ . Denote by  $\alpha$  the quantity that we want to compute. We have that

$$\alpha = \tau(u_{h_{m+1}}^* y_m^* \dots y_2^* u_{h_2}^* y_1^* u_{h_1}^{-1} g_1 x_1 u_{g_2} x_2 \dots x_n u_{g_{n+1}})$$

Assuming that  $\alpha \neq 0$ , let us prove that the first alternative holds.

Firstly, we must have that  $g_1 = h_1$  and  $i_1 = j_1$ , otherwise  $\alpha$  would be the trace of an alternating product. Hence  $x_1, y_1 \in M_{i_1} \ominus B$  and  $\alpha = \tau(u_{h_{m+1}}^* y_m^* \dots y_2^* u_{h_2}^* (y_1^* x_1) u_{g_2} x_2 \dots x_n u_{g_{n+1}})$ . Write  $y_1^* x_1 = b + z$ , where  $b \in B$  and  $z \in M_{i_1} \ominus B$ . Since  $u_{h_{m+1}}^* y_m^* \dots y_2^* u_{h_2}^* z u_{g_2} x_2 \dots x_n u_{g_{n+1}}$  is an alternating product and  $b$  commutes with  $\mathbb{F}_2$  we deduce that

$$\alpha = \tau(u_{h_{m+1}}^* y_m^* \dots y_2^* u_{h_2}^* b u_{g_2} x_2 \dots x_n u_{g_{n+1}}) = \langle u_{g_2} (b x_2) u_{g_3} \dots x_n u_{g_{n+1}}, u_{h_2} y_2 u_{h_3} \dots y_m u_{h_{m+1}} \rangle$$

By induction we get that  $n = m$ ,  $i_2 = j_2, \dots, i_n = j_n$  and that  $g_2 = h_2, \dots, g_n = h_n$ . It also follows that  $\alpha = \langle b x_2 x_3 \dots x_n, y_2 y_3 \dots y_n \rangle$ . Since the latter is equal to  $\langle x_1 x_2 \dots x_n, y_1 y_2 \dots y_n \rangle$ , we are done.  $\square$

Next, we present a crossed product decomposition of  $\tilde{M}$  (see [Io06, Remark 4.5]). Let  $N$  be the subalgebra of  $\tilde{M}$  generated by  $\{u_g M u_g^* | g \in \mathbb{F}_2\}$ . Then  $N$  is normalized by  $\mathbb{F}_2 = \{u_g\}_{g \in \mathbb{F}_2}$ . Since  $\tilde{M}$  is generated by  $N$  and  $\mathbb{F}_2$ , and  $E_N(u_g) = 0$ , for all  $g \in \mathbb{F}_2 \setminus \{e\}$ , we conclude that  $\tilde{M} = N \rtimes \mathbb{F}_2$ , where  $\mathbb{F}_2$  acts on  $N$  by conjugation.

Moreover, if  $\Sigma < \mathbb{F}_2$  is a subgroup, then for all  $g_1, g_2, \dots, g_{n+1} \in \mathbb{F}_2$  and every  $x_1, \dots, x_n \in M$ , we have that

$$(3.1) \quad E_{N \rtimes \Sigma}(u_{g_1} x_1 u_{g_2} x_2 \dots u_{g_n} x_n u_{g_{n+1}}) = \begin{cases} u_{g_1} x_1 u_{g_2} x_2 \dots u_{g_n} x_n u_{g_{n+1}}, & \text{if } g_1 g_2 \dots g_{n+1} \in \Sigma, \text{ and} \\ 0, & \text{if } g_1 g_2 \dots g_n g_{n+1} \notin \Sigma. \end{cases}$$

Note that the subalgebras  $\{u_g M u_g^*\}_{g \in \mathbb{F}_2}$  of  $\tilde{M}$  are freely independent over  $B$ . Therefore,  $N$  is isomorphic to the infinite amalgamated free product algebra  $M *_B * M *_B \dots$ . If we index the copies of  $M$  by  $\mathbb{F}_2$ , then the action of  $\mathbb{F}_2$  on  $N \cong M *_B * M *_B \dots$  is the *free Bernoulli shift*.

We are now ready to state the main result of this section.

**Theorem 3.2.** *Let  $A \subset pMp$  be a von Neumann subalgebra, for some projection  $p \in M$ .*

*Let  $t \in (0, 1)$ . Assume that  $\theta_t(A) \prec_{\tilde{M}} N$ . More generally, assume that  $\theta_t(A) \prec_{\tilde{M}} N \rtimes \Sigma$ , where  $\Sigma = \langle a \rangle$  is a cyclic subgroup of  $\mathbb{F}_2$ .*

*Then either  $A \prec_M B$  or  $\mathcal{N}_M(A)'' \prec_M M_i$ , for some  $i \in \{1, 2\}$ .*

Theorem 3.2 is an immediate consequence of Theorem 2.11 and the next lemma.



**Lemma 3.3.** *Let  $t \in (0, 1)$  and  $x_k \in (M)_1$  be a sequence such that  $\tau(\theta_t(x_k)x_k^*) \rightarrow 0$ .*

*Then  $\|E_N(y\theta_t(x_k)z)\|_2 \rightarrow 0$ , for every  $y, z \in \tilde{M}$ .*

*More generally, if  $\Sigma$  is a cyclic subgroup of  $\mathbb{F}_2$ , then  $\|E_{N \rtimes \Sigma}(y\theta_t(x_k)z)\|_2 \rightarrow 0$ , for every  $y, z \in \tilde{M}$ .*

*Proof of Theorem 3.2.* If  $\theta_t(A) \prec_{\tilde{M}} N \rtimes \Sigma$ , then by Theorem 2.1 we can find  $v \in \tilde{M}$  such that  $\inf_{u \in \mathcal{U}(A)} \|E_{N \rtimes \Sigma}(v\theta_t(u)v^*)\|_2 > 0$ . Lemma 3.3 then implies that  $\inf_{u \in \mathcal{U}(A)} \tau(\theta_t(u)u^*) > 0$ . Finally, the conclusion follows from Theorem 2.11.  $\square$

*Proof of Lemma 3.3.* Since  $\tilde{M} = N \rtimes \mathbb{F}_2$ , by Kaplansky's density theorem we may assume that  $y = u_g$  and  $z = u_h$ , for some  $g, h \in \mathbb{F}_2$ . Thus, our goal is to prove that  $\|E_{N \rtimes \Sigma}(u_g\theta_t(x_k)u_h)\|_2 \rightarrow 0$ . Let us first show that this is a consequence of the next lemma whose proof we postpone for now.

**Lemma 3.4.** *Fix  $t \in (0, 1)$  and for  $n \geq 0$ , define  $c_n = \sup_{x \in \mathcal{H}_n, \|x\|_2 \leq 1} \|E_{N \rtimes \Sigma}(u_g\theta_t(x)u_h)\|_2$ .*

*Then  $c_n \rightarrow 0$ , as  $n \rightarrow \infty$ .*

Assuming Lemma 3.4, let us finish the proof of Lemma 3.3. Write  $x_k = \sum_{n=0}^{\infty} x_{k,n}$ , with  $x_{k,n} \in \mathcal{H}_n$ . By equation 2.3 we have that  $\tau(\theta_t(x_k)x_k^*) = \sum_{n=0}^{\infty} (\frac{\sin(\pi t)}{\pi t})^{2n} \|x_{k,n}\|_2^2$ . Since  $\tau(\theta_t(x_k)x_k^*) \rightarrow 0$  and  $\sin(\pi t) > 0$ , we derive that  $\|x_{k,n}\|_2 \rightarrow 0$ , for all  $n \geq 0$ .

For  $n \geq 1$  and  $\mathcal{I} = (i_1, i_2, \dots, i_n) \in S_n$ , we let  $\mathcal{K}_{\mathcal{I}} \subset L^2(\tilde{M})$  be the closure of the linear span of

$$\{u_{h_1}x_1u_{h_2}x_2\dots u_{h_n}x_nu_{h_{n+1}} | h_1, \dots, h_{n+1} \in \mathbb{F}_2, x_1 \in M_{i_1} \ominus B, x_2 \in M_{i_2} \ominus B, \dots, x_n \in M_{i_n} \ominus B\}.$$

By Lemma 3.1 we have that if  $\mathcal{I} \in S_n$  and  $\mathcal{J} \in S_m$ , then  $\mathcal{K}_{\mathcal{I}} \perp \mathcal{K}_{\mathcal{J}}$ , unless  $n = m$  and  $\mathcal{I} = \mathcal{J}$ . Thus, denoting  $\mathcal{K}_n = \bigoplus_{\mathcal{I} \in S_n} \mathcal{K}_{\mathcal{I}}$ , we have that  $\mathcal{K}_n \perp \mathcal{K}_m$ , for all  $n \neq m$ .

By using the definition of  $\theta_t$  and equation 3.1 we derive that  $\theta_t(\mathcal{H}_{\mathcal{I}}) \subset \mathcal{K}_{\mathcal{I}}$  and  $E_{N \rtimes \Sigma}(\mathcal{K}_{\mathcal{I}}) \subset \mathcal{K}_{\mathcal{I}}$ . Since  $\mathcal{K}_{\mathcal{I}}$  is an  $L(\mathbb{F}_2)$ - $L(\mathbb{F}_2)$  bimodule, we deduce that  $E_{N \rtimes \Sigma}(u_g\theta_t(\mathcal{H}_{\mathcal{I}})u_h) \subset \mathcal{K}_{\mathcal{I}}$ . From this we get that  $E_{N \rtimes \Sigma}(u_g\theta_t(\mathcal{H}_n)u_h) \subset \mathcal{K}_n$ , for all  $n \geq 1$ .

Since the Hilbert spaces  $\{\mathcal{K}_n\}_{n \geq 1}$  are mutually orthogonal, the vectors  $\{E_{N \rtimes \Sigma}(u_g\theta_t(x_{k,n})u_h)\}_{n \geq 1}$  are mutually orthogonal, for all  $k \geq 1$ . By using this fact, the inequality  $\|\xi + \eta\|_2^2 \leq 2(\|\xi\|_2^2 + \|\eta\|_2^2)$  and the definition of  $c_n$ , we get that

$$\begin{aligned} \|E_{N \rtimes \Sigma}(u_g\theta_t(x_k)u_h)\|_2^2 &\leq 2\|E_{N \rtimes \Sigma}(u_g\theta_t(x_{k,0})u_h)\|_2^2 + 2\left\|\sum_{n=1}^{\infty} E_{N \rtimes \Sigma}(u_g\theta_t(x_{k,n})u_h)\right\|_2^2 = \\ &2\sum_{n=0}^{\infty} \|E_{N \rtimes \Sigma}(u_g\theta_t(x_{k,n})u_h)\|_2^2 \leq 2\sum_{n=0}^{\infty} c_n^2 \|x_{k,n}\|_2^2. \end{aligned}$$

Finally, let  $\varepsilon > 0$ . Since  $c_n \rightarrow 0$  by Lemma 3.4, we can find  $n_0 \geq 1$  such that  $c_n \leq \varepsilon$ , for all  $n \geq n_0$ . Since  $\|x_{k,n}\|_2 \rightarrow 0$ , for all  $n$ , we can also find  $k_0 \geq 1$  such that  $\|x_{k,i}\|_2 \leq \frac{\varepsilon}{n_0}$ , for all  $k \geq k_0$  and all  $i \in \{1, 2, \dots, n_0 - 1\}$ . Also, note that  $c_n \leq 1$ , for all  $n$ .

By using the above equation and the inequality  $\sum_{n=n_0}^{\infty} \|x_{k,n}\|_2^2 \leq \|x_k\|_2^2 = 1$ , it follows that

$$\|E_{N \rtimes \Sigma}(u_g\theta_t(x_k)u_h)\|_2^2 \leq 2(n_0(\frac{\varepsilon}{n_0})^2 + \varepsilon^2 \sum_{n=n_0}^{\infty} \|x_{k,n}\|_2^2) \leq 4\varepsilon^2, \text{ for all } k \geq k_0.$$

Since  $\varepsilon > 0$  was arbitrary, we are done.  $\square$

*Proof of Lemma 3.4.* For  $\mathcal{I} \in S_n$ , let  $c_{\mathcal{I}} = \sup_{x \in \mathcal{H}_{\mathcal{I}}, \|x\|_2 = 1} \|E_{N \rtimes \Sigma}(u_g\theta_t(x)u_h)\|_2$ . Recall that  $\mathcal{H}_n = \bigoplus_{\mathcal{I} \in S_n} \mathcal{H}_{\mathcal{I}}$ . Since  $u_g\theta_t(\mathcal{H}_{\mathcal{I}})u_h \subset \mathcal{K}_{\mathcal{I}}$  and the Hilbert spaces  $\{\mathcal{K}_{\mathcal{I}}\}_{\mathcal{I} \in S_n}$  are mutually orthogonal by Lemma 3.1, it follows that  $c_n = \max_{\mathcal{I} \in S_n} c_{\mathcal{I}}$ .

In the *first part of the proof*, we will find a formula for  $c_{\mathcal{I}}$ , for a fixed  $\mathcal{I} = (i_1, i_2, \dots, i_n) \in S_n$ .

Recall that  $a_1$  and  $a_2$  denote the generators of  $\mathbb{F}_2$ . Let  $G_1 = \langle a_1 \rangle$  and  $G_2 = \langle a_2 \rangle$  be the cyclic subgroups generated by  $a_1$  and  $a_2$ .

Let  $g_1, h_1 \in G_{i_1}$ ,  $g_2, h_2 \in G_{i_2}, \dots, g_n, h_n \in G_{i_n}$ . Then by Lemma 3.1, the map given by

$$(3.2) \quad V_{g_1, h_1, g_2, h_2, \dots, g_n, h_n}(x_1 x_2 \dots x_n) = u_{g_1} x_1 u_{h_1} u_{g_2} x_2 u_{h_2} \dots u_{g_n} x_n u_{h_n},$$

for all  $x_1 \in M_{i_1} \ominus B, x_2 \in M_{i_2} \ominus B, \dots, x_n \in M_{i_n} \ominus B$  extends to an isometry

$$V_{g_1, h_1, g_2, h_2, \dots, g_n, h_n} : \mathcal{H}_{\mathcal{I}} \rightarrow L^2(\tilde{M})$$

Moreover, Lemma 3.1 implies that  $V_{g_1, h_1, g_2, h_2, \dots, g_n, h_n}(\mathcal{H}_{\mathcal{I}}) \perp V_{g'_1, h'_1, g'_2, h'_2, \dots, g'_n, h'_n}(\mathcal{H}_{\mathcal{I}})$ , unless we have that  $g_1 = g'_1, h_1 g_2 = h'_1 g'_2, h_2 g_3 = h'_2 g'_3, \dots, h_{n-1} g_n = h'_{n-1} g'_n, h_n = h'_n$ . Since  $G_1 \cap G_2 = \{e\}$ , this implies that  $g_1 = g'_1, h_1 = h'_1, \dots, g_n = g'_n, h_n = h'_n$ .

Now, let  $\beta_1 : G_1 \rightarrow \mathbb{C}$  and  $\beta_2 : G_2 \rightarrow \mathbb{C}$  be given by  $\beta_1(g_1) = \tau(u_1^t u_{g_1}^*)$  and  $\beta_2(g_2) = \tau(u_2^t u_{g_2}^*)$ . Since  $u_1^t \in L(G_1)$  and  $u_2^t \in L(G_2)$ , we can decompose

$$(3.3) \quad u_1^t = \sum_{g_1 \in G_1} \beta_1(g_1) u_{g_1} \quad \text{and} \quad u_2^t = \sum_{g_2 \in G_2} \beta_2(g_2) u_{g_2}$$

where the sums converge in  $\|\cdot\|_2$ . Since  $u_1^t$  and  $u_2^t$  are unitaries, we have that

$$(3.4) \quad \sum_{g_1 \in G_1} |\beta_1(g_1)|^2 = \sum_{g_2 \in G_2} |\beta_2(g_2)|^2 = 1$$

If  $x = x_1 x_2 \dots x_n$ , for some  $x_1 \in M_{i_1} \ominus B, x_2 \in M_{i_2} \ominus B, \dots, x_n \in M_{i_n} \ominus B$ , then by 3.3 we have

$$\begin{aligned} u_g \theta_t(x) u_h &= u_g u_{i_1}^t x_1 u_{i_1}^{t*} u_{i_2}^t x_2 u_{i_2}^{t*} \dots u_{i_n}^t x_n u_{i_n}^{t*} u_h = \\ &\sum_{g_1, h_1 \in G_{i_1}, g_2, h_2 \in G_{i_2}, \dots, g_n, h_n \in G_{i_n}} \beta_{i_1}(g_1) \overline{\beta_{i_1}(h_1)} \beta_{i_2}(g_2) \overline{\beta_{i_2}(h_2)} \dots \beta_{i_n}(g_n) \overline{\beta_{i_n}(h_n)} u_g u_{g_1} x_1 u_{h_1} u_{g_2} x_2 u_{h_2} \dots u_{g_n} x_n u_{h_n} u_h \end{aligned}$$

By using equations 3.1 and 3.2, we further deduce that

$$(3.5) \quad \begin{aligned} E_{N \rtimes \Sigma}(u_g \theta_t(x) u_h) &= \\ \sum_{\substack{g_1, h_1 \in G_{i_1}, g_2, h_2 \in G_{i_2}, \dots, g_n, h_n \in G_{i_n} \\ g g_1 h_1 g_2 h_2 \dots g_n h_n h \in \Sigma}} \beta_{i_1}(g_1) \overline{\beta_{i_1}(h_1)} \beta_{i_2}(g_2) \overline{\beta_{i_2}(h_2)} \dots \beta_{i_n}(g_n) \overline{\beta_{i_n}(h_n)} u_g V_{g_1, h_1, g_2, h_2, \dots, g_n, h_n}(x) u_h. \end{aligned}$$

Since the linear span such elements  $x$  is dense in  $\mathcal{H}_{\mathcal{I}}$ , this formula holds for every  $x \in \mathcal{H}_{\mathcal{I}}$ . Since the isometries  $V_{g_1, h_1, g_2, h_2, \dots, g_n, h_n}$  have mutually orthogonal ranges, formula 3.5 implies that

$$\begin{aligned} \|E_{N \rtimes \Sigma}(u_g \theta_t(x) u_h)\|_2^2 &= \\ \|x\|_2^2 \sum_{\substack{g_1, h_1 \in G_{i_1}, g_2, h_2 \in G_{i_2}, \dots, g_n, h_n \in G_{i_n} \\ g g_1 h_1 g_2 h_2 \dots g_n h_n h \in \Sigma}} |\beta_{i_1}(g_1)|^2 |\beta_{i_1}(h_1)|^2 |\beta_{i_2}(g_2)|^2 |\beta_{i_2}(h_2)|^2 \dots |\beta_{i_n}(g_n)|^2 |\beta_{i_n}(h_n)|^2, \end{aligned}$$

for all  $x \in \mathcal{H}_{\mathcal{I}}$ .

Thus,

$$(3.6) \quad c_{\mathcal{I}} = \sum_{\substack{g_1, h_1 \in G_{i_1}, g_2, h_2 \in G_{i_2}, \dots, g_n, h_n \in G_{i_n} \\ g g_1 h_1 g_2 h_2 \dots g_n h_n h \in \Sigma}} |\beta_{i_1}(g_1)|^2 |\beta_{i_1}(h_1)|^2 |\beta_{i_2}(g_2)|^2 |\beta_{i_2}(h_2)|^2 \dots |\beta_{i_n}(g_n)|^2 |\beta_{i_n}(h_n)|^2$$

In the *second part of the proof*, we use this formula for  $c_{\mathcal{I}}$  to conclude that  $c_n \rightarrow 0$ . By 3.4 we can define probability measures  $\mu_1$  and  $\mu_2$  on  $\mathbb{F}_2$  by letting

$$(3.7) \quad \mu_i(g) = \begin{cases} |\beta_i(g)|^2, & \text{if } g \in G_i, \text{ and} \\ 0, & \text{if } g \notin G_i. \end{cases}$$

Denote  $\mu = \mu_1 * \mu_1 * \mu_2 * \mu_2$ . Then we have

**Claim.**  $\mu^{*n}(g\Sigma h) \rightarrow 0$ , for all  $g, h \in \mathbb{F}_2$ .

Assuming the claim, let us show that  $c_n \rightarrow 0$ . Firstly, the claim gives that  $(\nu_1 * \mu^{*n} * \nu_2)(g\Sigma h) \rightarrow 0$ , for any probability measures  $\nu_1, \nu_2$  on  $\mathbb{F}_2$  and all  $g, h \in \mathbb{F}_2$ . Secondly, the formula 3.6 rewrites as

$$c_{\mathcal{I}} = (\mu_{i_1} * \mu_{i_1} * \mu_{i_2} * \mu_{i_2} \dots * \mu_{i_n} * \mu_{i_n})(g^{-1}\Sigma h^{-1}).$$

Since  $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{n-1} \neq i_n$ , we have that  $\mu_{i_1} * \mu_{i_1} * \mu_{i_2} * \mu_{i_2} \dots * \mu_{i_n} * \mu_{i_n} \in \{\mu^{*[\frac{n}{2}]}, \mu^{*[\frac{n}{2}]} * \mu_1 * \mu_1, \mu_2 * \mu_2 * \mu^{*[\frac{n}{2}]}, \mu_2 * \mu_2 * \mu^{*[\frac{n-1}{2}]} * \mu_1 * \mu_1\}$ . By combining these facts it follows that  $c_n \rightarrow 0$ , as claimed.

*Proof of the claim.* Firstly, let us prove the claim in the case  $\Sigma = \{e\}$ . By Lemma 2.13 (1) it suffices to show that the support of  $\mu$  generates a non-amenable group.

Recall that  $u_{a_1} = \exp(i\alpha_1)$  and  $u_1^t = \exp(it\alpha_1)$ . Thus if  $n \in \mathbb{Z}$ , then

$$(3.8) \quad \mu_1(a_1^n) = |\tau(u_1^t u_{a_1}^*)|^2 = |\tau(u_1^{t-n})|^2 = \left(\frac{\sin(\pi(t-n))}{\pi(t-n)}\right)^2 = \frac{(\sin(\pi t))^2}{\pi^2(n-t)^2}.$$

Since  $t \in (0, 1)$ , it follows that  $\mu_1(a_1^n) \neq 0$  and similarly that  $\mu_2(a_2^n) \neq 0$ , for all  $n \in \mathbb{Z}$ . As a consequence the support of  $\mu$  contains  $a_1$  and  $a_2$ , and thus generates the whole  $\mathbb{F}_2$ .

In general, assume that  $\Sigma = \langle a \rangle$ , for some  $a \in \mathbb{F}_2$ . Let  $\ell : \mathbb{F}_2 \rightarrow \mathbb{N}$  be the word length on  $\mathbb{F}_2$  with respect to the generating set  $S = \{a_1, a_1^{-1}, a_2, a_2^{-1}\}$ . Note that 3.8 also implies that  $\mu_1(a_1^n) = \mu_2(a_2^n) \leq \frac{C}{|n|^{2+1}}$ , for all  $n \in \mathbb{Z}$ , where  $C = \frac{2}{t^2(1-t)^2}$ .

Let  $p \in (0, 1)$ . Since  $|i+j|^p \leq |i|^p + |j|^p$ , for  $i, j \geq 0$ , we get that

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |n|^p (\mu_1 * \mu_1)(a_1^n) &= \sum_{n \in \mathbb{Z}} |n|^p \left( \sum_{i+j=n} \mu_1(a_1^i) \mu_1(a_1^j) \right) \leq \\ C^2 \sum_{i, j \in \mathbb{Z}} \frac{|i|^p + |j|^p}{(|i|^2 + 1)(|j|^2 + 1)} &= 2C^2 \left( \sum_{i \in \mathbb{Z}} \frac{|i|^p}{|i|^2 + 1} \right) \left( \sum_{j \in \mathbb{Z}} \frac{1}{|j|^2 + 1} \right) < \infty. \end{aligned}$$

Now, the support of  $\mu$  is  $\{a_1^m a_2^n | m, n \in \mathbb{Z}\}$  and  $\ell(a_1^m a_2^n) = |m| + |n|$ , for every  $m, n \in \mathbb{Z}$ . By using the last inequality and the analogous one for  $\mu_2$  we derive that

$$\begin{aligned} \sum_{g \in \mathbb{F}_2} \ell(g)^p \mu(g) &= \sum_{m, n \in \mathbb{Z}} (|m| + |n|)^p (\mu_1 * \mu_1)(a_1^m) (\mu_2 * \mu_2)(a_2^n) \leq \\ \sum_{m \in \mathbb{Z}} |m|^p (\mu_1 * \mu_1)(a_1^m) &+ \sum_{n \in \mathbb{Z}} |n|^p (\mu_2 * \mu_2)(a_2^n) < \infty. \end{aligned}$$

Since  $\Sigma$  is a cyclic group, we can now apply Lemma 2.13 (2) to get the conclusion of the claim. This finishes the proof of the lemma.  $\square$

## 4. RELATIVE AMENABILITY AND SUBALGEBRAS OF AFP ALGEBRAS, I

Assume the notations from Sections 2.5 and 3. Thus,  $(M_1, \tau_1), (M_2, \tau_2)$  are tracial von Neumann algebras,  $M = M_1 *_B M_2$ ,  $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{F}_2))$  and  $N = \{u_g M u_g^* | g \in \mathbb{F}_2\}''$ .

Our goal in the next two sections is to understand what subalgebras  $A \subset M$  have the property that  $\theta_t(A)$  is amenable relative to  $N$ , for some (or all)  $t \in (0, 1)$ .

We start by considering the case  $A = M$ .

**Theorem 4.1.** *Suppose that  $M = M_1 *_B M_2$  is a factor and let  $p \in M$  be a projection.*

*If  $\theta_t(pMp)$  is amenable relative to  $N$  inside  $\tilde{M}$ , for some  $t \in (0, 1)$ , then either*

- (1)  $M_1 p_1$  is amenable relative to  $B$  inside  $M_1$ , for some non-zero projection  $p_1 \in \mathcal{Z}(M_1)$ , or
- (2)  $M_2 p_2$  is amenable relative to  $B$  inside  $M_2$ , for some non-zero projection  $p_2 \in \mathcal{Z}(M_2)$ .

In particular, if  $B$  is amenable and  $M_1, M_2$  have no amenable direct summands, then  $\theta_t(pMp)$  is not amenable relative to  $N$ , for any  $t \in (0, 1)$ . It would be interesting to determine whether the conclusion of Theorem 4.1 can be strengthened to “ $M$  is amenable relative to  $B$ ”.

In preparation for the proof of Theorem 4.1, we establish a useful decomposition of the  $M$ - $M$  bimodule  $L^2(\langle \tilde{M}, e_N \rangle)$ . Note that  $u_g M u_g^* \subset N$ , for all  $g \in \mathbb{F}_2$ . Equivalently,  $[u_g e_N u_g^*, M] = 0$ , for every  $g \in \mathbb{F}_2$ . Therefore,  $L^2(\langle \tilde{M}, e_N \rangle)$  contains an infinite direct sum of trivial  $M$ - $M$  bimodules:

$$\mathcal{H} = \bigoplus_{g \in \mathbb{F}_2} L^2(M) u_g e_N u_g^*.$$

If we let  $\mathcal{H}_2 = L^2(\langle \tilde{M}, e_N \rangle) \ominus \mathcal{H}$ , then we have the following

**Lemma 4.2.** *There is a  $B$ - $M$  bimodule  $\mathcal{K}$  such that  $\mathcal{H}_2 \cong L^2(M) \otimes_B \mathcal{K}$ , as  $M$ - $M$  bimodules.*

*Proof.* Since  $\tilde{M} = N \rtimes \mathbb{F}_2$ , we have that

$$L^2(\langle \tilde{M}, e_N \rangle) = \bigoplus_{g, h \in \mathbb{F}_2} L^2(N) u_g e_N u_h^*.$$

For  $g \in \mathbb{F}_2$ , let  $\sigma_g$  be the automorphism of  $N$  given by  $\sigma_g(x) = u_g x u_g^*$ , for  $x \in N$ . Then the  $N$ - $N$  bimodule  $L^2(N) u_g e_N u_h^*$  is isomorphic to  $L^2(N)$  endowed with the  $N$ - $N$  bimodule structure given by  $x \cdot \xi \cdot y = x \xi \sigma_{gh^{-1}}(y)$ , for all  $x, y \in N$  and  $\xi \in L^2(N)$ . For simplicity, we denote this bimodule by  ${}_N L^2(N)_{\sigma_{gh^{-1}}(N)}$ .

Next, we define the  $M$ - $M$  bimodules  $\mathcal{L} = L^2(N) \ominus L^2(M)$  and  $\mathcal{L}_g = {}_M L^2(N)_{\sigma_g(M)}$ . The first paragraph implies that  $\mathcal{H}_2 \cong \bigoplus_{i=1}^{\infty} (\mathcal{L} \oplus \bigoplus_{g \in \mathbb{F}_2 \setminus \{e\}} \mathcal{L}_g)$ , as  $M$ - $M$  bimodules.

Now, denote  $P = (\cup_{k \in \mathbb{F}_2 \setminus \{e\}} u_k M u_k^*)''$  and  $P_g = (\cup_{k \in \mathbb{F}_2 \setminus \{e, g\}} u_k M u_k^*)''$ , for  $g \in \mathbb{F}_2 \setminus \{e\}$ . Then  $N = M *_B P$  and  $N = M *_B \sigma_g(M) *_B P_g$ . By using Lemma 2.10 we can find a  $B$ - $M$  bimodule  $\mathcal{L}'$  and a  $B$ - $\sigma_g(M)$  bimodule  $\mathcal{L}'_g$  such that  $\mathcal{L} = L^2(M) \otimes_B \mathcal{L}'$  and  $\mathcal{L}_g = L^2(M) \otimes_B \mathcal{L}'_g$ , for all  $g \in \mathbb{F}_2 \setminus \{e\}$ . In combination with the last paragraph this yields the conclusion.  $\square$

In the proof of Theorem 4.1 we will also need a technical result showing that for  $t \in (0, 1)$ , the angle between the Hilbert spaces  $u_1^t \mathcal{H} u_1^{t*}$  and  $u_2^t \mathcal{H} u_2^{t*}$  is positive.

**Lemma 4.3.** *Let  $t \in (0, 1)$  and  $u_1^t, u_2^t \in L(\mathbb{F}_2)$  be the unitaries defined in Section 2.5. For  $i \in \{1, 2\}$ , we denote by  $P_i$  the orthogonal projection from  $L^2(\langle \tilde{M}, e_N \rangle)$  onto  $\mathcal{L}_i = u_i^t \mathcal{H} u_i^{t*}$ .*

*Then  $\|P_1 P_2\| < 1$ .*

*Proof.* Let  $S = P_1|_{\mathcal{L}_2} : \mathcal{L}_2 \rightarrow \mathcal{L}_1$ . Since  $\|P_1 P_2\| = \|S\|$  it suffices to prove that  $\|S\| < 1$ . We will achieve this by identifying  $S$  with the inflation of a certain contraction from  $L(\mathbb{F}_2)$ .

Given  $g \in \mathbb{F}_2$ , let  $\alpha_g = |\tau(u_1^{t*} u_2^t u_g^*)|^2$ . Note that  $\sum_{g \in \mathbb{F}_2} \alpha_g = 1$ . If we define the operator  $T = \sum_{g \in \mathbb{F}_2} \alpha_g \lambda(g) \in L(\mathbb{F}_2)$ , then it is clear that  $\|T\| \leq 1$ .

We claim that  $\|T\| < 1$ . To see this, recall that  $a_1$  and  $a_2$  are generators of  $\mathbb{F}_2$ . By using the same calculation as in 3.8 we get that  $u_1^t = \sum_{n \in \mathbb{Z}} \frac{\sin(\pi(t-n))}{\pi(t-n)} u_{a_1^n}$  and  $u_2^t = \sum_{n \in \mathbb{Z}} \frac{\sin(\pi(t-n))}{\pi(t-n)} u_{a_2^n}$ . It follows that  $\alpha_g \neq 0$  if and only if  $g \in \{a_1^m a_2^n | m, n \in \mathbb{Z}\}$ . Thus, the support of  $\alpha$  generates the whole  $\mathbb{F}_2$ . Since  $\mathbb{F}_2$  is non-amenable and  $\alpha_g \geq 0$ , for all  $g \in \mathbb{F}_2$ , we deduce that  $\|T\| < \sum_{g \in \mathbb{F}_2} \alpha_g = 1$ .

Next, for  $i \in \{1, 2\}$ , we define the unitary operator  $U_i : L^2(M) \bar{\otimes} \ell^2(\mathbb{F}_2) \rightarrow \mathcal{L}_i$  given by

$$U_i(\xi \otimes \delta_g) = u_i^t u_g \xi e_N u_g^* u_i^{t*}, \text{ for } \xi \in L^2(M) \text{ and } g \in \mathbb{F}_2.$$

Let  $g, h \in \mathbb{F}_2$ . Since  $u_h^* u_1^{t*} u_2^t u_g \in L(\mathbb{F}_2)$ , we get that  $E_N(u_h^* u_1^{t*} u_2^t u_g) = \tau(u_h^* u_1^{t*} u_2^t u_g) 1$ . Thus, for every  $\xi, \eta \in L^2(M)$  we get that

$$\begin{aligned} \langle U_1^* S U_2(\xi \otimes \delta_g), \eta \otimes \delta_h \rangle &= \langle P_1(u_2^t u_g \xi e_N u_g^* u_2^{t*}), u_1^t u_h \eta e_N u_h^* u_1^{t*} \rangle = \\ &= \langle u_2^t u_g \xi e_N u_g^* u_2^{t*}, u_1^t u_h \eta e_N u_h^* u_1^{t*} \rangle = |\tau(u_h^* u_1^{t*} u_2^t u_g)|^2 \langle \xi, \eta \rangle = \\ &= \alpha_{hg^{-1}} \langle \xi, \eta \rangle = \langle (1 \otimes T)(\xi \otimes \delta_g), \eta \otimes \delta_h \rangle. \end{aligned}$$

Therefore,  $S = U_1(1 \otimes T)U_2^*$  and since  $\|T\| < 1$  we get that  $\|S\| < 1$ .  $\square$

*Proof of Theorem 4.1.* Assume that  $\theta_t(pMp)$  is amenable relative to  $N$ , for some non-zero projection  $p \in M$ . Since  $M$  is a  $\text{II}_1$  factor it follows that  $\theta_t(M)$  is amenable relative to  $N$  (see Remark 2.2). By [OP07, Definition 2.2] we can find a net of vectors  $\xi_n \in L^2(\langle \tilde{M}, e_N \rangle)$  such that  $\langle x \xi_n, \xi_n \rangle \rightarrow \tau(x)$ , for all  $x \in \tilde{M}$ , and  $\|y \xi_n - \xi_n y\|_2 \rightarrow 0$ , for all  $y \in \theta_t(M)$ .

We denote  $\xi_{1,n} = u_1^{t*} \xi_n u_1^t$  and  $\xi_{2,n} = u_2^{t*} \xi_n u_2^t$ . Since  $\theta_t(y) = u_i^t y u_i^{t*}$ , for all  $y \in M_i$  and  $i \in \{1, 2\}$ , we derive that

$$(4.1) \quad \|y \xi_{1,n} - \xi_{1,n} y\| \rightarrow 0, \text{ for all } y \in M_1, \text{ and } \|y \xi_{2,n} - \xi_{2,n} y\| \rightarrow 0, \text{ for all } y \in M_2.$$

We also clearly have that

$$(4.2) \quad \langle x \xi_{1,n}, \xi_{1,n} \rangle \rightarrow \tau(x) \text{ and } \langle x \xi_{2,n}, \xi_{2,n} \rangle \rightarrow \tau(x), \text{ for all } x \in \tilde{M}$$

Denote by  $e$  and  $f$  the orthogonal projections from  $L^2(\langle \tilde{M}, e_N \rangle)$  onto  $\mathcal{H}_2 = L^2(\langle \tilde{M}, e_N \rangle) \ominus \mathcal{H}$  and onto  $\mathcal{H} = \bigoplus_{g \in \mathbb{F}_2} L^2(M) u_g e_N u_g^*$ , respectively. Since  $e + f = 1$ , we are in one of the following three cases:

**Case 1.**  $\limsup_n \|e(\xi_{1,n})\|_2 > 0$ .

**Case 2.**  $\limsup_n \|e(\xi_{2,n})\|_2 > 0$ .

**Case 3.**  $\|\xi_{1,n} - f(\xi_{1,n})\|_2 \rightarrow 0$  and  $\|\xi_{2,n} - f(\xi_{2,n})\|_2 \rightarrow 0$ .

In **Case 1**, since  $\mathcal{H}_2$  is a  $M$ - $M$  bimodule, equations 4.2 and 4.1 imply that  $\limsup_n \|x e(\xi_{1,n})\|_2 \leq \|x\|_2$ , for all  $x \in \tilde{M}$ , and  $\|y e(\xi_{1,n}) - e(\xi_{1,n}) y\|_2 \rightarrow 0$ , for all  $y \in M_1$ .

We claim that there is a  $B$ - $M_1$  bimodule  $\mathcal{K}_2$  such that  $\mathcal{H}_2 \cong L^2(M_1) \otimes_B \mathcal{K}_2$ , as  $M_1$ - $M_1$  bimodules. Assume for now that the claim holds. Then, since  $\limsup_n \|e(\xi_{1,n})\|_2 > 0$ , Lemma 2.3 implies that  $M_1 p_1$  is amenable relative to  $B$  inside  $M_1$ , for some non-zero projection  $p_1 \in \mathcal{Z}(M_1)$ .

Now, let us justify the claim. Firstly, Lemma 4.2 provides a  $B$ - $M$  bimodule  $\mathcal{K}$  such that  $\mathcal{H}_2 \cong L^2(M) \otimes_B \mathcal{K}$ , as  $M$ - $M$  bimodules. Since  $M = M_1 *_B M_2$ , by Lemma 2.10 we can find a  $B$ - $M_1$

bimodule  $\mathcal{K}_1$  such that  $L^2(M) \cong L^2(M_1) \otimes_B \mathcal{K}_1$ , as  $M_1$ - $M_1$  bimodules. Finally, it is clear that the  $B$ - $M_1$  bimodule  $\mathcal{K}_2 = \mathcal{K}_1 \otimes_B \mathcal{K}$  satisfies  $\mathcal{H}_2 \cong L^2(M_1) \otimes_B \mathcal{K}_2$ , as  $M_1$ - $M_1$  bimodules.

Similarly, in **Case 2**, we get that  $M_2 p_2$  is amenable relative to  $B$ , for a non-zero projection  $p_2 \in \mathcal{Z}(M_2)$ .

Finally, let us show that **Case 3** is impossible. Indeed, in this case we would have that  $\|\xi_n - u_1^t f(\xi_{1,n}) u_1^{t*}\|_2 \rightarrow 0$  and  $\|\xi_n - u_2^t f(\xi_{2,n}) u_2^{t*}\|_2 \rightarrow 0$ . Now, as in Lemma 4.3, for  $i \in \{1, 2\}$ , we let  $P_i$  be the orthogonal projection from  $L^2(\langle \tilde{M}, e_N \rangle)$  onto  $\mathcal{L}_i = u_i^t \mathcal{H} u_i^{t*}$ . Since  $u_i^t f(\xi_{i,n}) u_i^{t*} \in \mathcal{L}_i$ , we deduce that  $\|\xi_n - P_1(\xi_n)\|_2 \rightarrow 0$  and  $\|\xi_n - P_2(\xi_n)\|_2 \rightarrow 0$ .

Thus,  $\|\xi_n - P_1 P_2(\xi_n)\|_2 \rightarrow 0$ . On the other hand, Lemma 4.3 shows that  $\|P_1 P_2\| < 1$ . By combining these two facts we derive that  $\|\xi_n\|_2 \rightarrow 0$ , which is a contradiction.  $\square$

We end this section by noticing that Theorem 4.1 yields a particular case of Theorem 1.1:

*Proof of Theorem 1.1 in the case  $\Gamma_1$  and  $\Gamma_2$  are non-amenable, and  $\Lambda$  is amenable.* Therefore, let  $\Gamma \curvearrowright (X, \mu)$  be a free ergodic pmp action of  $\Gamma = \Gamma_1 *_{\Lambda} \Gamma_2$ . Recall that  $\cap_{i=1}^n g_i \Lambda g_i^{-1}$  is finite, for some  $g_1, g_2, \dots, g_n \in \Gamma$ , and denote  $M = L^\infty(X) \rtimes \Gamma$ .

We claim that any Cartan subalgebra  $A$  of  $M$  is unitarily conjugate to  $L^\infty(X)$ . To this end, notice that  $M = M_1 *_B M_2$ , where  $M_1 = L^\infty(X) \rtimes \Gamma_1$ ,  $M_2 = L^\infty(X) \rtimes \Gamma_2$  and  $B = L^\infty(X) \rtimes \Lambda$ . Let  $\tilde{M}$ ,  $\{\theta_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\tilde{M})$  and  $N$  be defined as above.

Let  $t \in (0, 1)$ . Since  $\tilde{M} = N \rtimes \mathbb{F}_2$ , by applying Theorem 2.8 to  $\theta_t(A) \subset \tilde{M}$  we have that either  $\theta_t(A) \prec_{\tilde{M}} N$  or  $\theta_t(M)$  is amenable relative to  $N$  inside  $\tilde{M}$ .

In the first case, Theorem 3.2 gives that either  $A \prec_M B = L^\infty(X) \rtimes \Lambda$  or  $M \prec_M M_i$ , for some  $i \in \{1, 2\}$ . If the first condition holds, then since  $M$  is a factor, [HPV10, Proposition 8] implies that  $A \prec_M L^\infty(X) \rtimes (\cap_{i=1}^n g_i \Lambda g_i^{-1})$ . Thus,  $A \prec_M L^\infty(X)$  and [Po01, Theorem A.1] gives that  $A$  and  $L^\infty(X)$  are indeed unitarily conjugate. On the other hand, the second condition cannot hold true. To see this, let  $g_1 \in \Gamma_1 \setminus \Lambda$  and  $g_2 \in \Gamma_2 \setminus \Lambda$ . Then the unitary  $u = u_{g_1 g_2}$  satisfies  $\|E_{M_i}(x u^n y)\|_2 \rightarrow 0$ , for every  $x, y \in M$ .

In the second case, Theorem 4.1 implies that  $M_i p_i$  is amenable relative to  $B$  for some  $p_i \in \mathcal{Z}(M_i)$  and some  $i \in \{1, 2\}$ . Since  $B$  is amenable, this would imply that  $M_i p_i$  is amenable. Since  $L(\Gamma_i) \subset M_i$  and  $\Gamma_i$  is non-amenable, this case is impossible.  $\square$

## 5. RELATIVE AMENABILITY AND SUBALGEBRAS OF AFP ALGEBRAS, II

Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be two tracial von Neumann algebras. Following the notations from Sections 2.5 and 3, we denote  $M = M_1 *_B M_2$ ,  $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{F}_2))$  and  $N = \{u_g M u_g^* | g \in \mathbb{F}_2\}''$ .

In this section we prove two structural results for subalgebras  $A \subset M$  with the property that  $\theta_t(A)$  is amenable relative to  $N$ , for any  $t \in (0, 1)$ . Firstly, we show:

**Theorem 5.1.** *Let  $A \subset pMp$  be a von Neumann subalgebra, for some projection  $p \in M$ . Let  $\omega$  be a free ultrafilter on  $\mathbb{N}$  and suppose that  $A' \cap (pMp)^\omega = \mathbb{C}p$ .*

*If  $\theta_t(A)$  is amenable relative to  $N$  inside  $\tilde{M}$ , for any  $t \in (0, 1)$ , then either*

- (1)  $A \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or
- (2)  $A$  is amenable relative to  $B$  inside  $M$ .

It seems to us that this theorem should hold without assuming that  $A' \cap (pMp)^\omega = \mathbb{C}p$ , but we were unable to prove this. This assumption is verified for instance if  $A = M$  and  $M$  is a  $\text{II}_1$  factor without property  $\Gamma$ . By [CH08, Corollary 3.2] if  $B$  is amenable and  $M_1$  is a  $\text{II}_1$  factor without

property  $\Gamma$ , then  $M = M_1 *_B M_2$  is a  $\text{II}_1$  factor which does not have property  $\Gamma$ . In the next section we will see more situations in which the above assumption holds.

Nevertheless, the condition  $A' \cap (pMp)^\omega = \mathbb{C}$  is not satisfied in other situations to which we would like to apply Theorem 5.1. For instance, let  $\Gamma = \Gamma_1 * \Gamma_2$  be a free product group and  $\Gamma \curvearrowright (X, \mu)$  be a free ergodic but not strongly ergodic action. Then the amalgamated free product  $\text{II}_1$  factor  $M = L^\infty(X) \rtimes \Gamma = (L^\infty(X) \rtimes \Gamma_1) *_B (L^\infty(X) \rtimes \Gamma_2)$  has property  $\Gamma$ .

In order to treat such situations, we prove the following variant of Theorem 5.1:

**Theorem 5.2.** *In the above setting, assume that we can decompose  $B = P \bar{\otimes} Q_0$ ,  $M_1 = P \bar{\otimes} Q_1$  and  $M_2 = P \bar{\otimes} Q_2$ , for some tracial von Neumann algebras  $P, Q_0, Q_1$  and  $Q_2$ . Note that  $M = P \bar{\otimes} Q$ , where  $Q = Q_1 *_B Q_2$ .*

*Let  $A \subset M$  be a von Neumann subalgebra. Suppose that there exist a subgroup  $\mathcal{U} \subset \mathcal{U}(P)$  and a homomorphism  $\rho : \mathcal{U} \rightarrow \mathcal{U}(Q)$  such that*

- $u \otimes \rho(u) \in A$ , for all  $u \in \mathcal{U}$ , and
- the von Neumann subalgebra  $A_0 \subset Q$  generated by  $\{\rho(u) | u \in \mathcal{U}\}$  satisfies  $A'_0 \cap Q^\omega = \mathbb{C}$ .

*If  $\theta_t(A)$  is amenable relative to  $N$  inside  $\tilde{M}$ , for any  $t \in (0, 1)$ , then either*

- (1)  $A_0 \prec_Q Q_i$ , for some  $i \in \{1, 2\}$ , or
- (2)  $A_0$  is amenable relative to  $Q_0$  inside  $Q$ .

In the rest of this section, we first prove Theorem 5.1 and then use it to deduce Theorem 5.2.

*Proof of Theorem 5.1.* Suppose by contradiction that conditions (1) and (2) fail. Recall that  $\mathcal{H} = \bigoplus_{g \in \mathbb{F}_2} L^2(M) u_g e_N u_g^*$  and  $\mathcal{H}_2 = L^2(\langle \tilde{M}, e_N \rangle) \ominus \mathcal{H}$ . We also define:

$$\mathcal{H}_0 = \bigoplus_{g \in \mathbb{F}_2} \mathbb{C} u_g e_N u_g^*, \quad \text{and} \quad \mathcal{H}_1 = \bigoplus_{g \in \mathbb{F}_2} (L^2(M) \ominus \mathbb{C}) u_g e_N u_g^*.$$

Note that  $L^2(\langle \tilde{M}, e_N \rangle) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ . For  $j \in \{0, 1, 2\}$ , we denote by  $e_j$  the orthogonal projection from  $L^2(\langle \tilde{M}, e_N \rangle)$  onto  $p\mathcal{H}_j p$ .

We denote by  $I$  the set of 4-tuples  $i = (X, Y, \delta, t)$  where  $X \subset \tilde{M}$  and  $Y \subset \mathcal{U}(A)$  are finite subsets,  $\delta \in (0, 1)$  and  $t \in (0, 1)$ . We make  $I$  a directed set by letting:  $(X, Y, \delta, t) \leq (X', Y', \delta', t')$  if and only if  $X \subset X', Y \subset Y', \delta' \leq \delta$  and  $t' \leq t$ .

Let  $i = (X, Y, \delta, t) \in I$ . Since  $\theta_t(A)$  is amenable relative to  $N$  inside  $\tilde{M}$ , by [OP07, Definition 2.2] we can find a vector  $\xi_i \in L^2(\langle \tilde{M}, e_N \rangle)$  such that

$$\begin{aligned} |\langle x \xi_i, \xi_i \rangle - \tau(x)| &\leq \delta, \quad \text{for all } x \in X, \\ |\langle (\theta_t(y) - y)^*(\theta_t(y) - y) \xi_i, \xi_i \rangle - \tau((\theta_t(y) - y)^*(\theta_t(y) - y))| &\leq \delta \quad \text{and} \\ \|\theta_t(y) \xi_i - \xi_i \theta_t(y)\|_2 &\leq \delta, \quad \text{for all } y \in Y. \end{aligned}$$

Moreover, following the proof of [OP07, Theorem 2.1] we may assume that  $\xi_i = \zeta_i^{\frac{1}{2}}$ , for some  $\zeta_i \in L^1(\langle \tilde{M}, e_N \rangle)_+$ . Thus,  $\langle x \xi_i, \xi_i \rangle = \text{Tr}(x \zeta_i) = \langle \xi_i x, \xi_i \rangle$ , for all  $x \in \tilde{M}$  and  $i \in I$ .

The first part of the proof consists of three claims.

**Claim 1.** We have that  $\langle x \xi_i, \xi_i \rangle \rightarrow \tau(x)$ , for all  $x \in \tilde{M}$ , and  $\|y \xi_i - \xi_i y\|_2 \rightarrow 0$ , for all  $y \in \mathcal{U}(A)$ .

*Proof of Claim 1.* The first assertion is clear. To prove the second assertion, let  $i = (X, Y, \delta, t) \in I$  and  $y \in Y$ . Then we have

$$\|(\theta_t(y) - y)\xi_i\|_2^2 = \langle (\theta_t(y) - y)^*(\theta_t(y) - y)\xi_i, \xi_i \rangle \leq \delta + \|\theta_t(y) - y\|_2^2.$$

Similarly, we have that  $\|\xi_i(\theta_t(y) - y)\|_2^2 \leq \delta + \|\theta_t(y) - y\|_2^2$ . By combining these inequalities we deduce that

$$\begin{aligned} \|y\xi_i - \xi_i y\|_2 &\leq \|\theta_t(y)\xi_i - \xi_i\theta_t(y)\|_2 + \|(\theta_t(y) - y)\xi_i\|_2 + \|\xi_i(\theta_t(y) - y)\|_2 \leq \\ &\delta + 2\sqrt{\delta + \|\theta_t(y) - y\|_2^2}. \end{aligned}$$

Since  $\|\theta_t(y) - y\|_2 \rightarrow 0$ , as  $t \rightarrow 0$ , it follows that  $\|y\xi_i - \xi_i y\|_2 \rightarrow 0$ .  $\square$

For  $i \in I$ , we denote  $\zeta_i = p\xi_i p \in pL^2(\langle \tilde{M}, e_N \rangle)p$ . Note that  $e_j(\xi_i) = e_j(\zeta_i)$ , for all  $j \in \{0, 1, 2\}$ .

**Claim 2.**  $\|\zeta_i - e_0(\zeta_i)\|_2 \rightarrow 0$ .

*Proof of Claim 2.* Since  $e_0(\zeta) + e_1(\zeta) + e_2(\zeta) = \zeta$ , for every  $\zeta \in pL^2(\langle \tilde{M}, e_N \rangle)p$ , it suffices to show that  $\|e_1(\zeta_i)\|_2 \rightarrow 0$  and  $\|e_2(\zeta_i)\|_2 \rightarrow 0$ .

Note first that  $p\mathcal{H}_1 p = \bigoplus_{g \in \mathbb{F}_2} (L^2(pMp) \ominus \mathbb{C}p)u_g e_N u_g^*$  is invariant under the map  $\xi \rightarrow y\xi y^*$ , for every  $y \in \mathcal{U}(pMp)$ . Thus, we have that  $e_1(y\xi_i y^*) = ye_1(\xi_i)y^*$  and further that

$$(5.1) \quad \begin{aligned} \|ye_1(\zeta_i)y^* - e_1(\zeta_i)\|_2 &= \|ye_1(\xi_i)y^* - e_1(\xi_i)\|_2 = \|e_1(y\xi_i y^* - \xi_i)\|_2 \leq \\ &\|y\xi_i y^* - \xi_i\|_2 \rightarrow 0, \text{ for all } y \in \mathcal{U}(A). \end{aligned}$$

On the other hand, since  $A' \cap (pMp)^\omega = \mathbb{C}p$ , Theorem 2.5 implies that we can find  $F \subset \mathcal{U}(A)$  finite and  $d > 0$  such that  $d\|\zeta\|_2^2 \leq \sum_{y \in F} \|y\zeta y^* - \zeta\|_2^2$ , for all  $\zeta \in L^2(pMp) \ominus \mathbb{C}p$ . This implies that

$$(5.2) \quad d\|\zeta\|_2^2 \leq \sum_{y \in F} \|y\zeta y^* - \zeta\|_2^2, \text{ for all } \zeta \in p\mathcal{H}_1 p.$$

The combination of 5.1 and 5.2 gives that  $\|e_1(\zeta_i)\|_2 \rightarrow 0$ , as claimed.

Next, since  $p\mathcal{H}_2 p$  is a  $pMp$ - $pMp$  bimodule,  $e_2$  is  $pMp$ - $pMp$  bimodular and therefore we have that

$$\begin{aligned} \limsup_i \|xe_2(\zeta_i)\|_2 &= \limsup_i \|xe_2(\xi_i)\|_2 = \limsup_i \|e_2(x\xi_i)\|_2 \leq \limsup_i \|x\xi_i\|_2 = \\ &\limsup_i \sqrt{\langle x^* x \xi_i, \xi_i \rangle} = \|x\|_2, \text{ for all } x \in M \end{aligned}$$

and that  $\|ye_2(\zeta_i) - e_2(\zeta_i)y\|_2 = \|e_2(y\xi_i - \xi_i y)\|_2 \leq \|y\xi_i - \xi_i y\|_2 \rightarrow 0$ , for all  $y \in \mathcal{U}(A)$ .

Now, recall that Lemma 4.2 shows that  $\mathcal{H}_2 \cong L^2(M) \otimes_B \mathcal{K}$ , for some  $B$ - $M$  bimodule  $\mathcal{K}$ . Thus, if  $\limsup_i \|e_2(\zeta_i)\|_2 > 0$ , then by Lemma 2.3 we could find a non-zero projection  $z \in \mathcal{Z}(A' \cap pMp)$  such that  $Az$  is amenable relative to  $B$  inside  $M$ . Since  $A' \cap pMp = \mathbb{C}$ , this would imply that  $A$  is amenable relative to  $B$  inside  $M$ , leading to a contradiction.  $\square$

Before proving our third claim, let us state two lemmas whose proofs we postpone for now. Denote by  $\lambda : \mathbb{F}_2 \rightarrow \mathcal{U}(\ell^2(\mathbb{F}_2))$  the left regular representation of  $\mathbb{F}_2$ . Then we have

**Lemma 5.3.** Define the unitary operator  $U : \mathcal{H}_0 \rightarrow \ell^2(\mathbb{F}_2)$  given by  $U(u_g e_N u_g^*) = \delta_g$ , for  $g \in \mathbb{F}_2$ .

If  $\eta \in \mathcal{H}_0$  and  $y \in \tilde{M}$ , then

$$\|y\eta - \eta y\|_2^2 = \sum_{g \in \mathbb{F}_2} \|\lambda(g)(U(\eta)) - U(\eta)\|^2 \|E_N(yu_g^*)\|_2^2.$$

**Lemma 5.4.** There exists  $c > 0$  such that if two elements  $g, h \in \mathbb{F}_2$  satisfy  $\|\lambda(g)(\eta) - \eta\| \leq c\|\eta\|$  and  $\|\lambda(h)(\eta) - \eta\| \leq c\|\eta\|$ , for some non-zero vector  $\eta \in \ell^2(\mathbb{F}_2)$ , then  $g$  and  $h$  commute.



Going back to the proof of Theorem 5.1, recall that Claim 2 yields that  $\|\zeta_i - e_0(\zeta_i)\|_2 \rightarrow 0$ . Moreover, Claim 1 gives that  $\|\zeta_i\|_2 \rightarrow \|p\|_2$  and that  $\|p\xi_i - \xi_i p\|_2 \rightarrow 0$ .

Thus, we can find  $i = (X, Y, \delta, t) \in I$  such that for every  $i' \geq i$  we have that

$$\|\zeta_{i'} - e_0(\zeta_{i'})\|_2 < \min\left\{\frac{c\|p\|_2}{128}, \frac{\|p\|_2}{4}\right\}, \quad \|\zeta_{i'}\|_2 \geq \frac{\|p\|_2}{2}, \quad \text{and} \quad \|p\xi_{i'} - \xi_{i'}p\|_2 \leq \frac{c\|p\|_2}{64}.$$

Note that  $\|p\theta_t(y)p\|_2 \geq \|p\|_2 - 2\|\theta_t(p) - p\|_2$ , for all  $y \in \mathcal{U}(p\tilde{M}p)$ . Since  $\lim_{t \rightarrow 0} \|\theta_t(p) - p\|_2 = 0$ , after eventually shrinking  $t$ , we may also assume that

$$(5.3) \quad \|p\theta_t(y)p\|_2 \geq \frac{\|p\|_2}{2}, \quad \text{for all } y \in \mathcal{U}(p\tilde{M}p)$$

Now, let  $i' \geq i$ . Then  $\|e_0(\zeta_{i'})\|_2 \geq \frac{\|p\|_2}{4}$ . Since  $e_0(\zeta_{i'}) \in p\mathcal{H}_0$ , we can write  $e_0(\zeta_{i'}) = \eta_{i'}p = p\eta_{i'}$ , for some  $\eta_{i'} \in \mathcal{H}_0$ . Then  $\|\eta_{i'}\|_2 = \frac{\|e_0(\zeta_{i'})\|_2}{\|p\|_2}$  and therefore  $\|\eta_{i'}\|_2 \geq \frac{1}{4}$ .

Also, we have that  $\|\zeta_{i'} - \xi_{i'}p\|_2 = \|p\xi_{i'}p - \xi_{i'}p\|_2 \leq \|p\xi_{i'} - \xi_{i'}p\|_2 \leq \frac{c\|p\|_2}{64}$  and similarly that  $\|\zeta_{i'} - p\xi_{i'}\|_2 \leq \frac{c\|p\|_2}{64}$ . By using these inequalities we derive the following

**Claim 3.** For every finite set  $F \subset \mathcal{U}(A)$  we can find a unit vector  $\eta \in \mathcal{H}_0$  depending on  $F$  such that

$$\|(p\theta_t(y)p)\eta - \eta(p\theta_t(y)p)\|_2 \leq \frac{c\|p\|_2}{4}, \quad \text{for all } y \in F.$$

*Proof of Claim 3.* Let  $i' = (X, Y \cup F, t, \min\{\delta, \frac{c\|p\|_2}{64}\})$  and define  $\eta := \frac{\eta_{i'}}{\|\eta_{i'}\|_2} \in \mathcal{H}_0$ .

Let  $y \in F$ . By the definition of  $\xi_{i'}$  we have that  $\|\theta_t(y)\xi_{i'} - \xi_{i'}\theta_t(y)\|_2 \leq \frac{c\|p\|_2}{64}$ . Since  $i' \geq i$ , by using the previous inequalities we derive that

$$(5.4) \quad \begin{aligned} \|(p\theta_t(y)p)\eta - \eta(p\theta_t(y)p)\|_2 &= \frac{1}{\|\eta_{i'}\|_2} \|p\theta_t(y)e_0(\zeta_{i'}) - e_0(\zeta_{i'})\theta_t(y)p\|_2 \leq \\ &4\|p\theta_t(y)\zeta_{i'} - \zeta_{i'}\theta_t(y)p\|_2 + 8\|\zeta_{i'} - e_0(\zeta_{i'})\|_2 \end{aligned}$$

Additionally, we have that

$$(5.5) \quad \begin{aligned} \|p\theta_t(y)\zeta_{i'} - \zeta_{i'}\theta_t(y)p\|_2 &\leq \|p\theta_t(y)\xi_{i'}p - p\xi_{i'}\theta_t(y)p\|_2 + \|\zeta_{i'} - \xi_{i'}p\|_2 + \|\zeta_{i'} - p\xi_{i'}\|_2 \leq \\ &\|\theta_t(y)\xi_{i'} - \xi_{i'}\theta_t(y)\|_2 + \frac{c\|p\|_2}{32} \leq \frac{3c\|p\|_2}{64}. \end{aligned}$$

Since  $\|\zeta_{i'} - e_0(\zeta_{i'})\|_2 \leq \frac{c\|p\|_2}{128}$ , by combining equations 5.4 and 5.5 the claim follows.  $\square$

In the *second part of the proof* we combine Lemmas 5.3, 5.4 and Claim 3 to get a contradiction. Since  $A \not\prec_{\tilde{M}} M_i$ , for all  $i \in \{1, 2\}$ , Theorem 3.2 implies that  $\theta_t(A) \not\prec_{\tilde{M}} N$  and moreover that  $\theta_t(A) \not\prec_{\tilde{M}} N \rtimes \Sigma$ , for any cyclic subgroup  $\Sigma < \mathbb{F}_2$ .

Thus, we can find  $y \in \mathcal{U}(A)$  such that  $\|E_N(p\theta_t(y)p)\|_2 \leq \frac{\|p\|_2}{4}$ . If we write  $p\theta_t(y)p = \sum_{g \in \mathbb{F}_2} y_g u_g$ , where  $y_g \in N$ , then  $\|y_e\|_2 \leq \frac{\|p\|_2}{4}$ . By applying Claim 3 to  $F = \{y\}$  we can find a unit vector  $\eta \in \mathcal{H}_0$  such that  $\|(p\theta_t(y)p)\eta - \eta(p\theta_t(y)p)\|_2 \leq \frac{c\|p\|_2}{4}$ .

Let  $S_1 = \{g \in \mathbb{F}_2 \mid \|\lambda(g)(U(\eta)) - U(\eta)\| > c\}$  and  $S_2 = \{g \in \mathbb{F}_2 \setminus \{e\} \mid \|\lambda(g)(U(\eta)) - U(\eta)\| \leq c\}$ . By using Lemma 5.3 we get that

$$\frac{c^2\|p\|_2^2}{16} \geq \|(p\theta_t(y)p)\eta - \eta(p\theta_t(y)p)\|_2^2 = \sum_{g \in \mathbb{F}_2} \|\lambda(g)(U(\eta)) - U(\eta)\|^2 \|y_g\|_2^2 \geq$$

$$c^2 \sum_{g \in S_1} \|y_g\|_2^2.$$

Hence, we derive that

$$(5.6) \quad \sum_{g \in S_1 \cup \{e\}} \|y_g\|_2^2 = \|y_e\|_2^2 + \sum_{g \in S_1} \|y_g\|_2^2 \leq \frac{\|p\|_2^2}{16} + \frac{\|p\|_2^2}{16} = \frac{\|p\|_2^2}{8}.$$

Since  $\sum_{g \in \mathbb{F}_2} \|y_g\|_2^2 = \|p\theta_t(y)p\|_2^2 \geq \frac{\|p\|_2^2}{4}$  by equation 5.3, we get that  $S_2 = \mathbb{F}_2 \setminus (S_1 \cup \{e\}) \neq \emptyset$ . On the other hand, by Lemma 5.4, any two elements  $g, h \in S_2$  commute. It follows that we can find  $k \in \mathbb{F}_2 \setminus \{e\}$  such that  $S_2 \subset \Sigma$ , where  $\Sigma = \{k^n | n \in \mathbb{Z}\}$ . Moreover, we can pick  $k$  such that if  $k' \in \mathbb{F}_2$  commutes with  $k^m$ , for some  $m \in \mathbb{Z} \setminus \{0\}$ , then  $k' \in \Sigma$ .

Further, since  $\theta_t(A) \not\prec_{\tilde{M}} N \rtimes \Sigma$ , we can find  $z \in \mathcal{U}(A)$  such that  $\|E_{N \rtimes \Sigma}(p\theta_t(z)p)\|_2 \leq \frac{\|p\|_2}{4}$ . Since  $y, z \in \mathcal{U}(A)$ , by applying Claim 3 to  $F = \{y, z\}$  we can find a unit vector  $\zeta \in \mathcal{H}_0$  such that  $\|(p\theta_t(y)p)\zeta - \zeta(p\theta_t(y)p)\|_2 \leq \frac{c\|p\|_2}{4}$  and  $\|(p\theta_t(z)p)\zeta - \zeta(p\theta_t(z)p)\|_2 \leq \frac{c\|p\|_2}{4}$ .

Let  $T_1 = \{g \in \mathbb{F}_2 | \|\lambda(g)(U(\zeta)) - U(\zeta)\| > c\}$  and  $T_2 = \{g \in \mathbb{F}_2 \setminus \{e\} | \|\lambda(g)(U(\zeta)) - U(\zeta)\| \leq c\}$ . Write  $p\theta_t(z)p = \sum_{g \in \mathbb{F}_2} z_g u_g$ , where  $z_g \in N$ . The same calculation as above then shows that

$$(5.7) \quad \sum_{g \in T_1} \|y_g\|_2^2 \leq \frac{\|p\|_2^2}{16} \quad \text{and} \quad \sum_{g \in T_1} \|z_g\|_2^2 \leq \frac{\|p\|_2^2}{16}$$

By combining inequalities 5.6 and 5.7 it follows that  $\sum_{g \in T_1 \cup (S_1 \cup \{e\})} \|y_g\|_2^2 \leq \frac{3\|p\|_2^2}{16}$ . Since we also have that  $\sum_{g \in \mathbb{F}_2} \|y_g\|_2^2 = \|p\theta_t(y)p\|_2^2 \geq \frac{\|p\|_2^2}{4}$ , we get that  $T_1 \cup S_1 \cup \{e\} \neq \mathbb{F}_2$ . Hence  $S_2 \cap T_2 \neq \emptyset$ .

Fix  $k' \in S_2 \cap T_2$ . If  $k'' \in T_2$ , then Lemma 5.4 implies that  $k''$  commutes with  $k'$ . Since  $k' \in S_2 \subset \Sigma \setminus \{e\}$ , we get that  $k'' \in \Sigma$  and therefore  $T_2 \subset \Sigma$ .

Thus,  $T_2 \cup \{e\} \subset \Sigma$  and so  $\sum_{g \in T_2 \cup \{e\}} \|z_g\|_2^2 \leq \|E_{N \rtimes \Sigma}(p\theta_t(z)p)\|_2^2 \leq \frac{\|p\|_2^2}{16}$ . Since  $T_1 \cup T_2 \cup \{e\} = \mathbb{F}_2$ , combining this inequality with 5.7 yields that  $\sum_{g \in \mathbb{F}_2} \|z_g\|_2^2 \leq \frac{\|p\|_2^2}{8}$ . This however contradicts the fact that  $\|p\theta_t(z)p\|_2 \geq \frac{\|p\|_2}{2}$  and finishes the proof.  $\square$

*Proof of Lemma 5.3.* Write  $\eta = \sum_{g \in \mathbb{F}_2} \eta_g u_g e_N u_g^*$ , where  $\eta_g \in \mathbb{C}$ , and  $y = \sum_{k \in \mathbb{F}_2} y_k u_k$ , where  $y_k \in N$ . Recall that the canonical semi-finite trace on  $\langle \tilde{M}, e_N \rangle$  is given by  $Tr(xe_N y) = \tau(xy)$ . If we denote by  $(\sigma_g)_{g \in \mathbb{F}_2}$  the conjugation action of  $\mathbb{F}_2$  on  $N$  (i.e.,  $\sigma_g(x) = u_g x u_g^*$ ), then we have

$$\begin{aligned} \langle y\eta, \eta y \rangle &= \sum_{g, h, k, l \in \mathbb{F}_2} \langle y_k u_k \eta_g u_g e_N u_g^*, \eta_h u_h e_N u_h^* y_l u_l \rangle = \\ &= \sum_{g, h, k, l \in \mathbb{F}_2} \eta_g \overline{\eta_h} Tr(y_k u_k u_g e_N u_g^* u_l^* u_h e_N u_h^*) = \\ &= \sum_{g, h, k, l \in \mathbb{F}_2} \eta_g \overline{\eta_h} \tau(E_N(u_h^* y_k u_k u_g) E_N(u_g^* u_l^* y_l^* u_h)). \end{aligned}$$

If  $g, k$  are fixed and the expression  $\tau(E_N(u_h^* y_k u_k u_g) E_N(u_g^* u_l^* y_l^* u_h))$  is non-zero, then  $h = gk$  and  $l = k$ . Moreover, in this case this expression is equal to  $\tau(\sigma_{(kg)^{-1}}(y_k) \sigma_{(kg)^{-1}}(y_k^*)) = \|y_k\|_2^2$ . Thus, we deduce that

$$\langle y\eta, \eta y \rangle = \sum_{g, k \in \mathbb{F}_2} \eta_g \overline{\eta_{kg}} \|y_k\|_2^2 = \sum_{k \in \mathbb{F}_2} \left( \sum_{g \in \mathbb{F}_2} \eta_{k^{-1}g} \overline{\eta_g} \right) \|y_k\|_2^2 =$$

$$\sum_{k \in \mathbb{F}_2} \langle \lambda(k)(U(\eta)), U(\eta) \rangle \|E_N(yu_k^*)\|_2^2.$$

Since we also have that  $\|y\eta\|_2 = \|\eta y\|_2 = \|y\|_2 \|\eta\|_2$ , the lemma follows.  $\square$

*Proof of Lemma 5.4.* Let  $a$  and  $b$  be generators of  $\mathbb{F}_2$ . Since  $\mathbb{F}_2$  is non-amenable, there exists  $c > 0$  any non-zero vector  $\eta \in \ell^2(\mathbb{F}_2)$  satisfies  $\|\lambda(a)(\eta) - \eta\|^2 + \|\lambda(b)(\eta) - \eta\|^2 > 2c^2 \|\eta\|^2$ .

Now, let  $g, h \in \mathbb{F}_2$  such that  $\|\lambda(g)(\eta) - \eta\| \leq c\|\eta\|$  and  $\|\lambda(h)(\eta) - \eta\| \leq c\|\eta\|$ , for some non-zero vector  $\eta \in \ell^2(\mathbb{F}_2)$ . From this we get that  $\|\lambda(g)(\eta) - \eta\|^2 + \|\lambda(h)(\eta) - \eta\|^2 \leq 2c^2 \|\eta\|^2$ .

Let  $\Delta < \mathbb{F}_2$  be the subgroup generated by  $g$  and  $h$ , and  $\gamma : \Delta \rightarrow \mathcal{U}(\ell^2(\Delta))$  be the its left regular representation. Since  $\mathbb{F}_2 = \sqcup_{g \in S} \Delta g$ , for a set  $S$  of representatives, the restriction  $\lambda|_\Delta$  is a subrepresentation of  $\oplus_{n=1}^\infty \gamma : \Delta \rightarrow \mathcal{U}(\oplus_{n=1}^\infty \ell^2(\Delta))$ . If we write  $\eta = (\eta_n)_{n=1}^\infty$ , where  $\eta_n \in \ell^2(\Delta)$ , then we can find  $n$  such that  $\|\gamma(g)(\eta_n) - \eta_n\|^2 + \|\gamma(h)(\eta_n) - \eta_n\|^2 \leq 2c^2 \|\eta_n\|^2$  and  $\eta_n \neq 0$ .

If  $g$  and  $h$  do not commute, then they generate a copy of  $\mathbb{F}_2$ . In other words, there exists an isomorphism  $\rho : \Delta \rightarrow \mathbb{F}_2$  such that  $\rho(g) = a$  and  $\rho(h) = b$ . In combination with the above, this leads to a contradiction.  $\square$

*Proof of Theorem 5.2.* Recall that  $B = P \bar{\otimes} Q_0$ ,  $M_1 = P \bar{\otimes} Q_1$  and  $M_2 = P \bar{\otimes} Q_2$ . Therefore,  $M = P \bar{\otimes} Q$ , where  $Q = Q_1 *_{Q_0} Q_2$ . Also, recall that  $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{F}_2))$  and that  $N = \{u_g e_M u_g^* | g \in \mathbb{F}_2\}''$ . We define  $\tilde{Q} = Q *_{Q_0} (Q_0 \bar{\otimes} L(\mathbb{F}_2))$  and  $N_0 = \{u_g Q u_g^* | g \in \mathbb{F}_2\}'' \subset \tilde{Q}$ . Note that  $\tilde{M} = P \bar{\otimes} \tilde{Q}$  and that  $N = P \bar{\otimes} N_0$ .

We denote by  $\{\alpha_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\tilde{Q})$  the free malleable deformation associated to the AFP decomposition  $Q = Q_1 *_{Q_0} Q_2$  (see Section 2.11). Then for every  $x \in P$  and  $y \in \tilde{Q}$  we have that  $\theta_t(x \otimes y) = x \otimes \alpha_t(y)$ .

Let  $t \in (0, 1)$ . We claim that  $\alpha_t(A_0)$  is amenable relative to  $N_0$  inside  $\tilde{Q}$ . Once this claim is proven the conclusion follows by applying Theorem 5.1 to the inclusion  $A_0 \subset Q = Q_1 *_{Q_0} Q_2$ .

Since  $\theta_t(A)$  is amenable relative to  $N$  inside  $\tilde{M}$ , by [OP07, Definition 2.2] we can find a  $\theta_t(A)$ -central state  $\Phi : \langle \tilde{M}, e_N \rangle \rightarrow \mathbb{C}$  such that  $\Phi|_{\tilde{M}} = \tau$ .

Since  $\tilde{M} = P \bar{\otimes} \tilde{Q}$  and that  $N = P \bar{\otimes} N_0$ , we have that  $\langle \tilde{M}, e_N \rangle = P \bar{\otimes} \langle \tilde{Q}, e_{N_0} \rangle$ . Define a state  $\Psi : \langle \tilde{Q}, e_{N_0} \rangle \rightarrow \mathbb{C}$  by  $\Psi(T) = \Phi(1 \otimes T)$  and let  $u \in \mathcal{U}$ . Since  $u \otimes \rho(u) \in A$  we have that  $u \otimes \alpha_t(\rho(u)) = \theta_t(u \otimes \rho(u)) \in \theta_t(A)$ . Thus for every  $T \in \langle \tilde{Q}, e_{N_0} \rangle$  we have that

$$\begin{aligned} \Psi(\alpha_t(\rho(u)) T \alpha_t(\rho(u))^*) &= \Phi(1 \otimes \alpha_t(\rho(u)) T \alpha_t(\rho(u))^*) = \\ \Phi((u \otimes \alpha_t(\rho(u)))(1 \otimes T)(u \otimes \alpha_t(\rho(u))^*)) &= \Phi(1 \otimes T) = \Psi(T). \end{aligned}$$

Thus,  $\Psi(\alpha_t(\rho(u)) T) = \Psi(T \alpha_t(\rho(u)))$ , for every  $u \in \mathcal{U}$  and  $T \in \langle \tilde{Q}, e_{N_0} \rangle$ . Since  $\{\alpha_t(\rho(u)) | u \in \mathcal{U}\}$  generates  $\alpha_t(A_0)$  and  $\Psi|_{\tilde{Q}} = \tau$ , we get that  $\Psi$  is  $\alpha_t(A_0)$ -central. Thus  $\alpha_t(A_0)$  is amenable relative to  $N_0$  inside  $\tilde{Q}$ . This proves the claim and finishes the proof.  $\square$

## 6. PROPERTY $\Gamma$ FOR SUBALGEBRAS OF AFP ALGEBRAS

Let  $Q$  be a von Neumann subalgebra of an amalgamated free product algebra  $M = M_1 *_B M_2$ . In this section we study the position of the relative commutant  $Q' \cap M^\omega$  inside  $M^\omega$ . We start by considering the case  $Q = M$ .

**Lemma 6.1.** *Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be tracial von Neumann algebras with a common von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B$ . Denote  $M = M_1 *_B M_2$ . Assume that there exist unitary elements  $u \in M_1$  and  $v, w \in M_2$  such that  $E_B(u) = E_B(v) = E_B(w) = E_B(w^*v) = 0$ .*

*If  $\omega$  is a free ultrafilter on  $\mathbb{N}$ , then  $M' \cap M^\omega \subset B^\omega$ .*

In the case  $B = \mathbb{C}1$  this result was proved in [Ba95, Theorem 11]. The proof of Theorem 6.1 is a straightforward adaptation of the proof of [Ba95, Theorem 11] to the case when  $B$  is arbitrary.

*Proof.* We denote by  $S_1 \subset M$  the set of alternating words in  $M_1 \ominus B$  and  $M_2 \ominus B$  that begin in  $M_1 \ominus B$ . Concretely,  $x \in S_1$  if we can write  $x = x_1 x_2 \dots x_n$ , for some  $x_1 \in M_1 \ominus B, x_2 \in M_2 \ominus B, x_3 \in M_1 \ominus B, \dots$ . Similarly, we denote by  $S_2 \subset M$  the set of alternating words in  $M_1 \ominus B$  and  $M_2 \ominus B$  that begin in  $M_2 \ominus B$ . For  $i \in \{1, 2\}$ , we denote by  $\mathcal{H}_i \subset L^2(M)$  the  $\|\cdot\|_2$  closure of the linear span of  $S_i$  and by  $P_i$  the orthogonal projection onto  $\mathcal{H}_i$ .

Note that if  $x \in M_1 \ominus B$  and  $y \in M_2 \ominus B$ , then  $x\mathcal{H}_2x^* \subset \mathcal{H}_1$  and  $y\mathcal{H}_1y^* \subset \mathcal{H}_2$ . The hypothesis therefore implies that

$$(6.1) \quad u\mathcal{H}_2u^* \subset \mathcal{H}_1, \quad v\mathcal{H}_1v^* \subset \mathcal{H}_2, \quad w\mathcal{H}_1w^* \subset \mathcal{H}_2 \text{ and } v\mathcal{H}_1v^* \perp w\mathcal{H}_1w^*$$

The last fact holds because  $(w^*v)\mathcal{H}_1(w^*v)^* \subset \mathcal{H}_2$  and hence  $(w^*v)\mathcal{H}_1(w^*v)^* \perp \mathcal{H}_1$ .

Now, let  $\xi \in L^2(M)$ . Notice that if  $P_{\mathcal{K}}$  is the orthogonal projection onto a closed subspace  $\mathcal{K} \subset L^2(M)$  and  $u \in \mathcal{U}(M)$ , then  $P_{u\mathcal{K}u^*}(\xi) = uP_{\mathcal{K}}(u^*\xi u)u^*$  and therefore  $\|P_{u\mathcal{K}u^*}(\xi)\|_2 = \|P_{\mathcal{K}}(u^*\xi u)\|_2$ . By combining this fact with equation 6.1 we get that

$$(6.2) \quad \|P_2(u^*\xi u)\|_2 \leq \|P_1(\xi)\|_2 \text{ and } \|P_1(v^*\xi v)\|_2^2 + \|P_1(w^*\xi w)\|_2^2 \leq \|P_2(\xi)\|_2^2$$

Let  $x = (x_n)_n \in M' \cap M^\omega$ . Then  $\|u^*x_n u - x_n\|_2, \|v^*x_n v - x_n\|_2, \|w^*x_n w - x_n\|_2 \rightarrow 0$ , as  $n \rightarrow \omega$ . Using this fact and applying 6.2 to  $\xi = x_n$  we get that  $\lim_{n \rightarrow \omega} \|P_2(x_n)\|_2 \leq \lim_{n \rightarrow \omega} \|P_1(x_n)\|_2$  and  $\sqrt{2} \lim_{n \rightarrow \omega} \|P_1(x_n)\|_2 \leq \lim_{n \rightarrow \omega} \|P_2(x_n)\|_2$ . Therefore, we have that  $\|P_1(x_n)\|_2 \rightarrow 0$  and  $\|P_2(x_n)\|_2 \rightarrow 0$ , as  $n \rightarrow \omega$ .

Since  $L^2(M) = L^2(B) \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ , it follows that  $\lim_{n \rightarrow \omega} \|x_n - E_B(x_n)\|_2 = 0$  and thus  $x \in B^\omega$ .  $\square$

Lemma 6.1 implies that a large class of AFP groups give rise to  $\text{II}_1$  factors without property  $\Gamma$ .

**Corollary 6.2.** *Let  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  be an amalgamated free product group such that  $[\Gamma_1 : \Lambda] \geq 2$  and  $[\Gamma_2 : \Lambda] \geq 3$ . Assume that there exist  $g_1, g_2, \dots, g_m \in \Gamma$  such that  $\cap_{i=1}^m g_i \Lambda g_i^{-1} = \{e\}$ .*

*Then  $L(\Gamma)$  is a  $\text{II}_1$  factor without property  $\Gamma$ .*

*Moreover,  $\Gamma$  is not inner amenable, i.e. the unitary representation  $\pi : \Gamma \rightarrow \mathcal{U}(\ell^2(\Gamma \setminus \{e\}))$  given by  $\pi(g)(\delta_h) = \delta_{ghg^{-1}}$ , for  $g \in \Gamma$  and  $h \in \Gamma \setminus \{e\}$ , does not have almost invariant vectors.*

*Proof.* Let  $x = (x_n)_n \in L(\Gamma)' \cap L(\Gamma)^\omega$ . Firstly, by Lemma 6.1 we get that  $x \in L(\Lambda)^\omega$ .

Secondly, for  $i \in \{1, 2, \dots, m\}$ , denote by  $E_i$  the conditional expectation onto  $L(g_i \Lambda g_i^{-1})$ . Then  $E_i(x) = u_{g_i} E_{L(\Lambda)}(u_{g_i}^* x u_{g_i}) u_{g_i}^*$ , for every  $x \in L(\Gamma)$ . Since  $(x_n)_n \in L(\Gamma)' \cap L(\Lambda)^\omega$  it follows that  $\|E_i(x_n) - x_n\|_2 \rightarrow 0$ , as  $n \rightarrow \omega$ , for every  $i \in \{1, 2, \dots, m\}$ .

On the other hand, since  $\cap_{i=1}^m g_i \Lambda g_i^{-1} = \{e\}$ , we derive that  $E_1 E_2 \dots E_m(x) = \tau(x)1$ , for all  $x \in L(\Gamma)$ . Altogether, it follows that  $\|\tau(x_n)1 - x_n\|_2 \rightarrow 0$ , as  $n \rightarrow \omega$ , i.e.  $(x_n)_n \in \mathbb{C}1$ .

We leave it the reader to modify the above proof to show that  $\Gamma$  is indeed non-inner amenable.  $\square$

Next, we show that if a von Neumann subalgebra  $Q \subset M = M_1 *_B M_2$  is “large” (i.e. if conditions (2) and (3) below are not satisfied) then a corner of  $Q' \cap M^\omega$  embeds into  $B^\omega$ . Thus, the phenomenon from Theorem 6.1 extends in some sense to arbitrary subalgebras  $Q \subset M$ .

**Theorem 6.3.** *Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be tracial von Neumann algebras with a common von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B$ . Let  $M = M_1 *_B M_2$  and  $Q \subset pMp$  be a von Neumann subalgebra, for some projection  $p \in M$ . Let  $\omega$  be a free ultrafilter on  $\mathbb{N}$ . Denote by  $P$  the von Neumann subalgebra of  $M^\omega$  generated by  $M$  and  $B^\omega$ .*

*Then one of the following conditions holds true:*

- (1)  $Q' \cap (pMp)^\omega \subset P$  and  $Q' \cap (pMp)^\omega \prec_P B^\omega$ .
- (2)  $\mathcal{N}_{pMp}(Q)'' \prec_M M_i$ , for some  $i \in \{1, 2\}$ .
- (3)  $Qp'$  is amenable relative to  $B$ , for some non-zero projection  $p' \in \mathcal{Z}(Q' \cap pMp)$ .

To prove Theorem 6.3 we will need the following result.

**Theorem 6.4.** [CH08] *Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be tracial von Neumann algebras with a common von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B$ . Let  $M = M_1 *_B M_2$  and  $Q \subset pMp$  be a von Neumann subalgebra, for some projection  $p \in M$ .*

*Then one of the following conditions holds:*

- (1)  $Q' \cap pMp \prec_M B$ .
- (2)  $\mathcal{N}_{pMp}(Q)'' \prec_M M_i$ , for some  $i \in \{1, 2\}$ .
- (3)  $Qp'$  is amenable relative to  $B$ , for some non-zero projection  $p' \in \mathcal{Z}(Q' \cap pMp)$ .

In the case when  $B$  is amenable and  $Q$  has no amenable direct summand this result was proved by I. Chifan and C. Houdayer [CH08, Theorem 1.1]. The argument that we include below follows closely their proof.

Note that part (1) of Theorem 6.3 implies part (1) of Theorem 6.4. Indeed, if  $Q' \cap (pMp)^\omega \prec_P B^\omega$ , then  $(Q' \cap pMp)^\omega \prec_{M^\omega} B^\omega$ . This readily implies that  $Q' \cap M \prec_M B$ . Therefore Theorem 6.3 is stronger than Theorem 6.4.

Before proceeding to the proofs of Theorems 6.3 and 6.4, let us fix some notations. Let  $\tilde{M} = M * (B \bar{\otimes} L(\mathbb{F}_2))$  and  $\{\theta_t\}_{t \in \mathbb{R}}$  be the automorphisms of  $\tilde{M}$  defined in Section 2.11. We extend  $\theta_t$  to an automorphism of  $\tilde{M}^\omega$  by putting  $\theta_t((x_n)_n) = (\theta_t(x_n))_n$ . For  $x \in \tilde{M}^\omega$ , we denote

$$\delta_t(x) = \theta_t(x) - E_{M^\omega}(\theta_t(x)) \in \tilde{M}^\omega \ominus M^\omega.$$

Note that if  $x \in \tilde{M}$ , then  $\delta_t(x) \in \tilde{M} \ominus M$ .

Let  $\beta$  be the automorphism of  $\tilde{M}$  satisfying  $\beta(x) = x$  if  $x \in M$ ,  $\beta(u_{a_1}) = u_{a_1}^*$  and  $\beta(u_{a_2}) = u_{a_2}^*$ , where  $a_1, a_2$  are the generators of  $\mathbb{F}_2$  chosen in Section 2.11. We still denote by  $\beta$  the extension of  $\beta$  to  $\tilde{M}^\omega$ . It is easy to check that  $\beta^2 = id_{\tilde{M}^\omega}$  and  $\beta\theta_t\beta = \theta_{-t}$ , for all  $t \in \mathbb{R}$ .

By [Po06a, Lemma 2.1], the existence of  $\beta$  implies that

$$(6.3) \quad \|\theta_{2t}(x) - x\|_2 \leq 2\|\delta_t(x)\|_2, \quad \text{for all } x \in M \text{ and every } t \in \mathbb{R}.$$

In the proofs of Theorems 6.3 and 6.4 we assume for simplicity that  $p = 1$ , the general case being treated similarly. We continue with the following lemma which is key in both proofs.

**Lemma 6.5.** *Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be tracial von Neumann algebras with a common von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B$ . Let  $Q \subset M = M_1 *_B M_2$  be a von Neumann subalgebra such that  $Qp'$  is not amenable relative to  $B$ , for any non-zero projection  $p' \in \mathcal{Z}(Q' \cap M)$ .*

*Then we have that  $\sup_{x \in (Q' \cap M^\omega)_1} \|\delta_t(x)\|_2 \rightarrow 0$ , as  $t \rightarrow 0$ .*

*Proof.* It is easy to see that the map  $\mathbb{R} \ni t \rightarrow \|\delta_t(x)\|_2 \in [0, \infty)$  is even on  $\mathbb{R}$ , and decreasing on  $[0, \infty)$ , for every  $x \in \tilde{M}^\omega$ . Thus, if the lemma is false, then there exists  $c > 0$  such that  $\sup_{x \in (Q' \cap M^\omega)_1} \|\delta_t(x)\|_2 > c$ , for every  $t \in \mathbb{R} \setminus \{0\}$ .

For  $m \geq 1$ , put  $t_m = 2^{-m}$ . Let  $x_m \in (Q' \cap M^\omega)_1$  such that  $\xi_m = \delta_{t_m}(x_m)$  satisfies  $\|\xi_m\|_2 > c$ .

Fix  $y \in M$  and  $z \in (Q)_1$ . Then we have that

$$\|y\xi_m\|_2 = \|(1 - E_{M^\omega})(y\theta_{t_m}(x_m))\|_2 \leq \|y\theta_{t_m}(x_m)\|_2 \leq \|y\|_2.$$

Also, since  $zx_m = x_mz$ , by using S. Popa's spectral gap argument [Po06b] we get that

$$\begin{aligned} \|z\xi_m - \xi_mz\|_2 &= \|(1 - E_M)(z\theta_{t_m}(x_m) - \theta_{t_m}(x_m)z)\|_2 \leq \|z\theta_{t_m}(x_m) - \theta_{t_m}(x_m)z\|_2 = \\ &= \|\theta_{-t_m}(z)x_m - x_m\theta_{-t_m}(z)\|_2 \leq 2\|\theta_{-t_m}(z) - z\|_2 \rightarrow 0. \end{aligned}$$

By writing  $\xi_m = (\xi_{m,n})_n$ , where  $\xi_{m,n} \in \tilde{M} \ominus M$ , we find a net  $\eta_k \in \tilde{M} \ominus M$  such that  $\|\eta_k\|_2 > c$ ,  $\limsup_k \|y\eta_k\|_2 \leq \|y\|_2$ , for every  $y \in M$ , and  $\|z\eta_k - \eta_kz\|_2 \rightarrow 0$ , for every  $z \in Q$ .

Now, since  $\tilde{M} = M * (B \bar{\otimes} L(\mathbb{F}_2))$ , by Lemma 2.10 we have that  $L^2(\tilde{M}) \ominus L^2(M) \cong L^2(M) \otimes_B \mathcal{K}$ , for some  $B$ - $M$  bimodule  $\mathcal{K}$ . We may therefore apply Lemma 2.3 to conclude that  $Qp'$  is amenable relative to  $B$ , for a non-zero projection  $p' \in \mathcal{Z}(Q' \cap M)$ , which gives a contradiction.  $\square$

*Proof of Theorem 6.4.* Assuming that condition (3) is false, we prove that either (1) or (2) holds.

Since  $Q' \cap M \subset Q' \cap M^\omega$ , Lemma 6.5 implies that  $\sup_{x \in (Q' \cap M)_1} \|\delta_t(x)\|_2 \rightarrow 0$ , as  $t \rightarrow 0$ . Together with inequality 6.3 this yields  $t > 0$  such that  $\|\theta_t(x) - x\|_2 \leq \frac{1}{2}$ , for all  $x \in (Q' \cap M)_1$ .

Thus,  $\tau(\theta_t(u)u^*) \geq \frac{1}{2}$ , for every  $u \in \mathcal{U}(Q' \cap M)$ . Applying Theorem 2.11 gives that either  $Q' \cap M \prec_M B$  or  $\mathcal{N}_M(Q' \cap M)'' \prec_M M_i$ , for some  $i \in \{1, 2\}$ . Since  $\mathcal{N}_M(Q) \subset \mathcal{N}_M(Q' \cap M)$ , this finishes the proof.  $\square$

In the proof of Theorem 6.3 we will also use the following technical result:

**Lemma 6.6.** *Let  $\tilde{P}$  be the von Neumann subalgebra of  $\tilde{M}^\omega$  generated by  $\tilde{M}$  and  $B^\omega$ .*

*Then we have*

- (1)  $M_1^\omega$  and  $M_2^\omega$  are freely independent over  $B^\omega$ ,
- (2)  $M^\omega \perp (\tilde{P} \ominus P)$  and
- (3)  $(\tilde{M} \ominus M)(M^\omega \ominus P) \perp M^\omega(\tilde{M} \ominus M)$ .

*Proof.* Let  $x_1 \in M_{i_1}^\omega \ominus B^\omega, x_2 \in M_{i_2}^\omega \ominus B^\omega, \dots, x_m \in M_{i_m}^\omega \ominus B^\omega$ , for some indices  $i_1, i_2, \dots, i_m \in \{1, 2\}$  such that  $i_k \neq i_{k+1}$ , for all  $1 \leq k \leq m-1$ . Then we can represent  $x_k = (x_{k,n})_n$ , where  $x_{k,n} \in M_{i_k} \ominus B$ , for all  $n$  and every  $1 \leq k \leq m$ . Since  $\tau_\omega(x_1 x_2 \dots x_m) = \lim_{n \rightarrow \omega} \tau(x_{1,n} x_{2,n} \dots x_{m,n}) = 0$ , the first assertion follows.

Towards the second assertion, define  $P_1 = \{M_1, B^\omega\}''$ ,  $P_2 = \{M_2, B^\omega\}''$  and  $P_3 = \{B \bar{\otimes} L(\mathbb{F}_2), B^\omega\}''$ . All of these algebras contain  $B^\omega$  and we have that  $P_1 \subset M_1^\omega$ ,  $P_2 \subset M_2^\omega$  and  $P_3 \subset (B \bar{\otimes} L(\mathbb{F}_2))^\omega$ . Now, the first assertion implies that  $M_1^\omega, M_2^\omega$  and  $(B \bar{\otimes} L(\mathbb{F}_2))^\omega$  are freely independent over  $B^\omega$ . Since  $P = \{P_1, P_2\}''$  and  $\tilde{P} = \{P_1, P_2, P_3\}''$ , we deduce that

$$\tilde{P} \ominus P \subset (M_1^\omega *_{B^\omega} M_2^\omega *_{B^\omega} (B \bar{\otimes} L(\mathbb{F}_2))^\omega) \ominus (M_1^\omega *_{B^\omega} M_2^\omega).$$

It is clear that the latter space is orthogonal to  $M^\omega$ , thereby proving (2).

Finally, let  $z_1, z_2 \in \tilde{M} \ominus M$ ,  $y_1 \in M^\omega \ominus P$  and  $y_2 \in M^\omega$  such that  $\|y_1\|, \|y_2\| \leq 1$ . Write  $y_1 = (y_{1,n})_n, y_2 = (y_{2,n})_n$ , where  $y_{1,n}, y_{2,n} \in (M)_1$ . Our goal is to prove that  $\langle z_1 y_1, y_2 z_2 \rangle = 0$  or, equivalently, that  $\lim_{n \rightarrow \omega} \langle z_1 y_{1,n}, y_{2,n} z_2 \rangle = 0$ .

Since  $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{F}_2))$ , by Lemma 2.10 we can find a  $M$ - $B$  bimodule  $\mathcal{K}$  such that  $L^2(\tilde{M}) \ominus L^2(M) = \mathcal{K} \otimes_B L^2(M)$ . Viewing  $z_1, z_2$  as vectors in  $L^2(\tilde{M}) \ominus L^2(M)$  and using approximations in  $\|\cdot\|_2$ , we may assume that  $z_1 = \xi_1 \otimes_B \eta_1$ ,  $z_2 = \xi_2 \otimes_B \eta_2$ , where  $\xi_1, \xi_2 \in \mathcal{K}$  and  $\eta_1, \eta_2 \in M$ . Moreover, we may take  $\xi_1$  to be right bounded, i.e. such that  $\|\xi_1 y\|_2 \leq C\|y\|_2$ , for all  $y \in M$ , for some constant  $C > 0$ . By using the definition of Connes' tensor product we get that

$$|\langle z_1 y_{1,n}, y_{2,n} z_2 \rangle| = |\langle y_{2,n}^* \xi_1 \otimes_B \eta_1 y_{1,n}, \xi_2 \otimes_B \eta_2 \rangle| = |\langle y_{2,n}^* \xi_1 E_B(\eta_1 y_{1,n} \eta_2^*), \xi_2 \rangle| \leq C \|E_B(\eta_1 y_{1,n} \eta_2^*)\|_2 \|\xi_2\|_2.$$

Since  $y_1 \perp P$  and  $\eta_1^* B^\omega \eta_2 \subset P$ , we get that  $y_1 \perp \eta_1^* B^\omega \eta_2$ . Hence,  $\lim_{n \rightarrow \omega} \|E_B(\eta_1 y_{1,n} \eta_2^*)\|_2 = \|E_{B^\omega}(\eta_1 y_1 \eta_2^*)\|_2 = 0$ , which proves the last assertion.  $\square$

To prove Theorem 6.3 we adapt the proof of [Io10, Lemma 3.3] (see also the proof of [Bo12, Theorem 3.8]) to the case of AFP algebras. In the proof of Theorem 6.3 we apply Theorem 6.4 and [IPP05, Theorems 1.1 and 3.1] to *non-separable* tracial von Neumann algebras. While these results are only stated for separable algebras, their proofs can be easily modified to handle non-separable algebras. We leave the details to the reader.

*Proof of Theorem 6.3.* For simplicity, we assume that  $p = 1$ . Assuming that (2) and (3) are false, we will deduce that (1) holds. The proof is divided between two claims, each proving one assertion from (1).

**Claim 1.**  $Q' \cap M^\omega \subset P$ .

*Proof of Claim 1.* Assume by contradiction that there exists  $x \in Q' \cap M^\omega$  such that  $x \notin P$  and put  $y = x - E_P(x) \neq 0$ . Fix  $z \in (Q)_1$  and  $t \in \mathbb{R}$ .

Since  $E_{M^\omega}(\theta_t(z)) = (E_{M^\omega} \circ E_{\tilde{M}})(\theta_t(z)) = E_M(\theta_t(z))$  and  $y \in M^\omega$  we get that

$$(6.4) \quad \begin{aligned} \|\delta_t(z)y - y\delta_t(z)\|_2 &= \|(1 - E_M)(\theta_t(z))y - y(1 - E_M)(\theta_t(z))\|_2 = \\ &= \|(1 - E_{M^\omega})(\theta_t(z)y - y\theta_t(z))\|_2 \leq \|\theta_t(z)y - y\theta_t(z)\|_2 \end{aligned}$$

Since  $zx = xz$  and  $z \in M \subset P$ , we get that  $zy = yz$ . Thus, we derive that

$$(6.5) \quad \|\theta_t(z)y - y\theta_t(z)\|_2 = \|z\theta_{-t}(y) - \theta_{-t}(y)z\|_2 \leq 2\|\theta_{-t}(y) - y\|_2 = 2\|\theta_t(y) - y\|_2$$

On the other hand, since  $x \in M^\omega$ , Lemma 6.6 (2) gives that  $E_{\tilde{P}}(x) = E_P(x)$ . Since  $\theta_t$  leaves  $\tilde{P}$  globally invariant we conclude that  $\theta_t(E_P(x)) = \theta_t(E_{\tilde{P}}(x)) = E_{\tilde{P}}(\theta_t(x))$ . As a consequence, we have

$$(6.6) \quad \|\theta_t(y) - y\|_2 = \|(1 - E_{\tilde{P}})(\theta_t(x) - x)\|_2 \leq \|\theta_t(x) - x\|_2$$

By combining 6.4, 6.5 and 6.6 we get that  $\|\delta_t(z)y - y\delta_t(z)\|_2 \leq 2\|\theta_t(x) - x\|_2$ .

Since  $\delta_t(z) \in \tilde{M} \ominus M$  and  $y \in M^\omega \ominus P$ , Lemma 6.6 (3) implies that  $\delta_t(z)y \perp y\delta_t(z)$ . Therefore we derive that  $\|\delta_t(z)y\|_2 \leq 2\|\theta_t(x) - x\|_2$ . Since

$$\|\delta_t(z)y - \delta_t(zy)\|_2 \leq \|\theta_t(z)y - \theta_t(zy)\|_2 \leq \|\theta_t(y) - y\|_2,$$

we altogether deduce that  $\|\delta_t(zy)\|_2 \leq 3\|\theta_t(x) - x\|_2$ , for every  $z \in (Q)_1$  and  $t \in \mathbb{R}$ .

By using this inequality together with 6.3 and 6.6 we derive that

$$(6.7) \quad \begin{aligned} \|\theta_t(z)y - zy\|_2 &\leq \|\theta_t(zy) - zy\|_2 + \|\theta_t(y) - y\|_2 \leq \\ &2\|\delta_{\frac{t}{2}}(zy)\|_2 + \|\theta_t(y) - y\|_2 \leq 6\|\theta_{\frac{t}{2}}(x) - x\|_2 + \|\theta_t(x) - x\|_2 \leq \end{aligned}$$

$$12\|\delta_{\frac{t}{4}}(x)\|_2 + 2\|\delta_{\frac{t}{2}}(x)\|_2, \text{ for all } z \in (Q)_1 \text{ and } t \in \mathbb{R}.$$

Now, since (3) is assumed false, Lemma 6.5 implies that  $\sup_{x \in (Q' \cap M^\omega)_1} \|\delta_t(x)\|_2 \rightarrow 0$ , as  $t \rightarrow 0$ . In combination with 6.7 it follows that we can find  $t > 0$  such that  $\|\theta_t(z)y - zy\|_2 \leq \frac{\|y\|_2}{2}$ , for all  $z \in (Q)_1$ . Thus, if we let  $w = E_{\tilde{M}}(yy^*)$ , then

$$\Re \tau(\theta_t(z)wz^*) = \Re \tau(\theta_t(z)yy^*z^*) \geq \frac{\|y\|_2^2}{2}, \text{ for all } z \in \mathcal{U}(Q).$$

By using a standard averaging argument we can find  $0 \neq v \in \tilde{M}$  such that  $\theta_t(z)v = vz$ , for all  $z \in Q$ . By [IPP05, Theorem 3.1] we would conclude that  $Q \prec_M M_i$ , for some  $i \in \{1, 2\}$ .

If we denote  $\mathcal{N} = \mathcal{N}_M(Q)''$ , then [IPP05, Theorem 1.1] would imply that either  $\mathcal{N} \prec_M M_1$ ,  $\mathcal{N} \prec_M M_2$  or  $Q \prec_M B$ . Since the last condition implies that there is a non-zero projection  $p' \in \mathcal{Z}(Q' \cap M)$  such that  $Qp'$  is amenable relative to  $B$ , we altogether get a contradiction.  $\square$

To end the proof we are left with showing:

**Claim 2.**  $Q' \cap M^\omega \prec_P B^\omega$ .

*Proof of Claim 2.* Recall from the proof of Lemma 6.6 that  $P_1 = \{M_1, B^\omega\}''$  and  $P_2 = \{M, B^\omega\}''$  are freely independent over  $B^\omega$ , and that  $P = P_1 *_B P_2$ .

By applying Theorem 6.4 to the inclusion  $Q \subset P$  it follows that we are in one of the following three cases: (a)  $Q' \cap P \prec_P B^\omega$ , (b)  $\mathcal{N}_P(Q)'' \prec_P P_i$ , for some  $i \in \{1, 2\}$ , or (c)  $Qz$  is amenable relative to  $B^\omega$  inside  $P$ , for some non-zero projection  $z \in \mathcal{Z}(Q' \cap P)$ .

In case (a), Claim 1 implies that  $Q' \cap M^\omega = Q' \cap P \prec_P B^\omega$  and thus (1) is satisfied. Let us show that cases (b) and (c) contradict our assumption that conditions (2) and (3) are false.

Firstly, since  $\mathcal{N} \subset \mathcal{N}_P(Q)''$ ,  $P_i \subset M_i^\omega$  and  $P \subset M^\omega$ , case (b) implies that  $\mathcal{N} \prec_{M^\omega} M_i^\omega$ . By Remark 2.2 it follows that  $\mathcal{N}p_0$  is amenable relative to  $M_i^\omega$  inside  $M^\omega$ , for some non-zero projection  $p_0 \in \mathcal{N}' \cap M^\omega$ . Lemma 2.4 further implies that  $\mathcal{N}p'$  is amenable relative to  $M_i$  inside  $M$ , for some non-zero projection  $p' \in \mathcal{N}' \cap M$ . By Corollary 2.12 we get that either (b<sub>1</sub>)  $\mathcal{N}p'$  is amenable relative to  $B$  inside  $M$  or (b<sub>2</sub>)  $\mathcal{N} \prec_M M_i$ . In the case (b<sub>1</sub>) we get in particular that  $Qp''$  is amenable relative to  $B$  inside  $M$ , contradicting the assumption that (3) is false. In turn, case (b<sub>2</sub>) contradicts the assumption that (2) does not hold.

Finally, in case (c), Lemma 2.4 implies that  $Qp'$  is amenable relative to  $B$ , for some non-zero projection  $p' \in \mathcal{Z}(Q' \cap M)$ . In other words, (3) holds, a contradiction.  $\square$

## 7. UNIQUENESS OF CARTAN SUBALGEBRAS FOR $\text{II}_1$ FACTORS ARISING FROM ACTIONS OF AFP GROUPS

The main goal of this section is to prove Theorem 1.1 and derive several consequences.

**7.1. Uniqueness of Cartan subalgebras.** Towards proving Theorem 1.1 we first establish a general technical result.

**Theorem 7.1.** *Let  $\Gamma_1$  and  $\Gamma_2$  be two countable groups with a common subgroup  $\Lambda$  such that  $[\Gamma_1 : \Lambda] \geq 2$  and  $[\Gamma_2 : \Lambda] \geq 3$ . Denote  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  and suppose that there exist  $g_1, g_2, \dots, g_n \in \Gamma$  such that  $\cap_{i=1}^n g_i \Lambda g_i^{-1}$  is finite.*

*Let  $\Gamma \curvearrowright (D, \tau)$  be any trace preserving action of  $\Gamma$  on a tracial von Neumann algebra  $(D, \tau)$ . Denote  $M = D \rtimes \Gamma$  and suppose that  $M$  is a factor.*

*If  $A$  is a regular amenable von Neumann subalgebra of  $M$ , then  $A \prec_M D$ .*



Before proceeding to the proof of Theorem 7.1, let us introduce some notations that will essentially allow us to reduce to the case when  $\cap_{i=1}^n g_i \Lambda g_i^{-1}$  is trivial and not only finite.

Since  $\cap_{i=1}^n g_i \Lambda g_i^{-1}$  is finite,  $\Sigma = \cap_{g \in \Gamma} g \Lambda g^{-1}$  is a finite group and there exist  $h_1, h_2, \dots, h_m \in \Gamma$  such that  $\Sigma = \cap_{j=1}^m h_j \Lambda h_j^{-1}$ . Since  $\Sigma < \Lambda$  is a normal subgroup of  $\Gamma$ , we can define the following groups  $\Gamma' = \Gamma/\Sigma$ ,  $\Gamma'_1 = \Gamma_1/\Sigma$ ,  $\Gamma'_2 = \Gamma_2/\Sigma$  and  $\Lambda' = \Lambda/\Sigma$ . Note that  $\Gamma' = \Gamma'_1 *_{\Lambda'} \Gamma'_2$  and let  $\rho : \Gamma \rightarrow \Gamma'$  be the quotient homomorphism. Note also that  $\cap_{j=1}^m k_j \Lambda' k_j^{-1} = \{e\}$ , where  $k_j = \rho(h_j)$ .

Denote  $\mathcal{M} = M \bar{\otimes} L(\Gamma')$  and let  $\Delta : M \rightarrow \mathcal{M}$  be the *comultiplication* [PV09] defined by

$$\Delta(a u_g) = a u_g \otimes u_{\rho(g)}, \quad \text{for every } a \in D \text{ and all } g \in \Gamma.$$

We next record a property of  $\Delta$  that will be of later use.

**Lemma 7.2.** *Let  $Q \subset M$  be a von Neumann subalgebra and  $\Gamma_0 < \Gamma$  be a subgroup.*

*If  $\Delta(Q) \prec_{\mathcal{M}} M \bar{\otimes} L(\rho(\Gamma_0))$ , then  $Q \prec_M D \rtimes \Gamma_0$ .*

*Proof of Lemma 7.2.* Assume by contradiction that  $Q \not\prec_M D \rtimes \Gamma_0$ . Then we can find a sequence of unitaries  $u_n \in Q$  such that  $\|E_{D \rtimes \Gamma_0}(x u_n y)\|_2 \rightarrow 0$ , for all  $x, y \in M$ . We claim that  $\|E_{M \bar{\otimes} L(\rho(\Gamma_0))}(v \Delta(u_n) w)\|_2 \rightarrow 0$ , for all  $v, w \in \mathcal{M}$ . This will provide the desired contradiction.

To prove the claim, by Kaplansky's density theorem, we may assume that  $v = 1 \otimes u_{\rho(h)}$  and  $w = 1 \otimes u_{\rho(k)}$ , for some  $h, k \in \Gamma$ . For every  $n$ , write  $u_n = \sum_{g \in \Gamma} x_{n,g} u_g$ , where  $x_{n,g} \in D$ . Then  $\Delta(u_n) = \sum_{g \in \Gamma} x_{n,g} u_g \otimes u_{\rho(g)}$ . Since  $\ker(\rho) = \Sigma$ , it follows that

$$E_{M \bar{\otimes} L(\rho(\Gamma_0))}(v \Delta(u_n) w) = \sum_{g \in \Gamma} x_{n,g} u_g \otimes E_{L(\rho(\Gamma_0))}(u_{\rho(h g k)}) = \sum_{g \in h^{-1} \Gamma_0 \Sigma k^{-1}} x_{n,g} u_g \otimes u_{\rho(h g k)}.$$

Further, since  $\Sigma$  is finite we deduce that

$$\|E_{M \bar{\otimes} L(\rho(\Gamma_0))}(v \Delta(u_n) w)\|_2^2 = \sum_{g \in h^{-1} \Gamma_0 \Sigma k^{-1}} \|x_{n,g}\|_2^2 \leq \sum_{l \in \Sigma} \|E_{D \rtimes \Gamma_0}(u_h u_n u_{kl})\|_2^2.$$

Since  $\|E_{D \rtimes \Gamma_0}(u_h u_n u_{kl})\|_2 \rightarrow 0$ , as  $n \rightarrow \infty$ , the lemma is proven.  $\square$

*Proof of Theorem 7.1.* Define  $\mathcal{M}_1 = M \bar{\otimes} L(\Gamma'_1)$ ,  $\mathcal{M}_2 = M \bar{\otimes} L(\Gamma'_2)$  and  $B = M \bar{\otimes} L(\Lambda')$ . Then we have that  $\mathcal{M} = \mathcal{M}_1 *_B \mathcal{M}_2$ .

Define  $\tilde{\mathcal{M}} = \mathcal{M} *_B (B \bar{\otimes} L(\mathbb{F}_2))$  and let  $\{\theta_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\tilde{\mathcal{M}})$  be the deformation defined in Section 2.11. Also, let  $N$  be the von Neumann subalgebra of  $\tilde{\mathcal{M}}$  generated by  $\{u_g \mathcal{M} u_g^* | g \in \mathbb{F}_2\}$ . Recall from Section 3 that  $\tilde{\mathcal{M}} = N \rtimes \mathbb{F}_2$ , where  $\mathbb{F}_2 = \{u_g\}_{g \in \mathbb{F}_2}$  acts on  $N$  by conjugation.

Let  $t \in (0, 1)$  and consider the amenable von Neumann subalgebra  $\theta_t(\Delta(A)) \subset \tilde{\mathcal{M}}$ . By S. Popa and S. Vaes' dichotomy (Theorem 2.8) we get that either  $\theta_t(\Delta(A)) \prec_{\tilde{\mathcal{M}}} N$  or  $\mathcal{N}_{\tilde{\mathcal{M}}}(\theta_t(\Delta(A)))''$  is amenable relative to  $N$  inside  $\tilde{\mathcal{M}}$ .

Since  $A$  is regular in  $M$ , we have that  $\theta_t(\Delta(M)) \subset \mathcal{N}_{\tilde{\mathcal{M}}}(\theta_t(\Delta(A)))''$ . Therefore, we are in one of the following two cases:

**Case 1.** There exists  $t \in (0, 1)$  such that  $\theta_t(\Delta(A)) \prec_{\tilde{\mathcal{M}}} N$ .

**Case 2.** For every  $t \in (0, 1)$  we have that  $\theta_t(\Delta(M))$  is amenable relative to  $N$  inside  $\tilde{\mathcal{M}}$ .

In **Case 1**, Theorem 3.2 gives that either  $\Delta(A) \prec_{\mathcal{M}} B$  or  $\mathcal{N}_{\mathcal{M}}(\Delta(A))'' \prec_{\mathcal{M}} \mathcal{M}_i$ , for some  $i \in \{1, 2\}$ . Since  $A$  is regular in  $M$ , the latter condition implies that  $\Delta(M) \prec_{\mathcal{M}} \mathcal{M}_i$ .

By using Lemma 7.2 we derive that either  $A \prec_M D \rtimes \Lambda$  or  $M \prec_M D \rtimes \Gamma_i$ , for some  $i \in \{1, 2\}$ . If  $A \prec_M D \rtimes \Lambda$ , then as  $M$  is a factor, [HPV10, Proposition 8] implies that  $A \prec_M D \rtimes (\cap_{i=1}^n g_i \Lambda g_i^{-1})$ . Since  $\cap_{i=1}^n g_i \Lambda g_i^{-1}$  is finite, we conclude that  $A \prec_M D$ , as claimed.

Now, since  $[\Gamma_1 : \Lambda] \geq 2$  and  $[\Gamma_2 : \Lambda] \geq 2$ , we can find  $g_1 \in \Gamma_1 \setminus \Lambda$  and  $g_2 \in \Gamma_2 \setminus \Lambda$ . Let  $u = u_{g_1 g_2} \in \mathcal{U}(L(\Gamma))$ . Then we have that  $\|E_{D \rtimes \Gamma_i}(xu^n y)\|_2 \rightarrow 0$ , for every  $x, y \in M$  and  $i \in \{1, 2\}$ . Thus,  $L(\Gamma) \not\prec_M D \rtimes \Gamma_i$  and hence  $M \not\prec_M D \rtimes \Gamma_i$ . This shows that the second alternative is impossible and finishes the proof of **Case 1**.

In **Case 2**, since  $[\Gamma'_1 : \Lambda'] \geq 2$ ,  $[\Gamma'_2 : \Lambda'] \geq 3$  and  $\cap_{j=1}^m k_j \Lambda' k_j^{-1} = \{e\}$ , Corollary 6.2 implies that  $L(\Gamma')' \cap L(\Gamma')^\omega = \mathbb{C}1$ .

Note that  $u_g \otimes u_{\rho(g)} \in \Delta(M)$ , for every  $g \in \Gamma$ . Moreover, the von Neumann algebra  $A_0$  generated by  $\{u_{\rho(g)}\}_{g \in \Gamma}$  is equal to  $L(\Gamma')$  and satisfies  $A'_0 \cap L(\Gamma')^\omega = \mathbb{C}1$ . Since  $\theta_t(\Delta(M))$  is amenable relative to  $N$ , for any  $t \in (0, 1)$ , by Theorem 5.2 we deduce that either  $L(\Gamma') \prec_{L(\Gamma')} L(\Gamma'_i)$ , for some  $i \in \{1, 2\}$ , or  $L(\Gamma')$  is amenable relative to  $L(\Lambda')$  inside  $L(\Gamma')$ .

Since  $[\Gamma'_1 : \Lambda'] \geq 2$  and  $[\Gamma'_2 : \Lambda'] \geq 2$ , we can choose  $g_1 \in \Gamma'_1 \setminus \Lambda'$  and  $g_2 \in \Gamma'_2 \setminus \Lambda'$ . Then  $u = u_{g_1 g_2} \in L(\Gamma')$  satisfies  $\|E_{L(\Gamma'_1)}(xu^n y)\|_2 \rightarrow 0$  and  $\|E_{L(\Gamma'_2)}(xu^n y)\|_2 \rightarrow 0$ , for all  $x, y \in L(\Gamma')$ , showing that the first alternative is impossible.

Finally, if  $L(\Gamma')$  is amenable relative to  $L(\Lambda')$  inside  $L(\Gamma')$ , then  $\Lambda'$  is *co-amenable* in  $\Gamma'$ , i.e. there exists a  $\Gamma'$ -invariant state  $\Phi : \ell^\infty(\Gamma'/\Lambda') \rightarrow \mathbb{C}$  (see [AD95, Proposition 3.5]). Let us show that is impossible as well.

Let  $g_1 \in \Gamma'_1 \setminus \Lambda'$  and  $g_2, g_3 \in \Gamma'_2 \setminus \Lambda'$  such that  $g_3^{-1} g_2 \notin \Lambda'$ . Let  $S_1$  and  $S_2$  be the set of words in  $\Gamma'_1 \setminus \Lambda'$  and  $\Gamma'_2 \setminus \Lambda'$  beginning in  $\Gamma'_1 \setminus \Lambda'$  and in  $\Gamma'_2 \setminus \Lambda'$ , respectively. Then  $\Gamma' = S_1 \sqcup S_2 \sqcup \Lambda'$  and we have  $\Lambda' \subset g_1 S_1, g_1 S_2 \subset S_1, g_2 S_1 \subset S_2, g_3 S_1 \subset S_2$ .

Now, let  $q : \Gamma' \rightarrow \Gamma'/\Lambda'$  be quotient map and define  $T_1 = q(S_1), T_2 = q(S_2)$ . Then we have  $\Gamma'/\Lambda' = T_1 \sqcup T_2 \sqcup \{e\Lambda'\}$  and  $e\Lambda' \in g_1 T_1, g_1 T_2 \subset T_1, g_2 T_1 \subset T_2, g_3 T_1 \subset T_2$ . Moreover, since  $g_3^{-1} g_2 T_1 \subset T_2$ , we get that  $g_2 T_1 \cap g_3 T_1 = \emptyset$ . Hence,  $g_2 T_1 \sqcup g_3 T_1 \subset T_2$ .

For a subset  $T \subset \Gamma'/\Lambda'$ , let  $m(T) = \Phi(1_T) \in [0, 1]$ . Then  $m$  is a finitely additive  $\Gamma'$ -invariant probability measure on  $\Gamma'/\Lambda'$ . The relations from the last paragraph therefore imply that  $m(e\Lambda') \leq m(T_1), m(T_2) \leq m(T_1)$  and  $2m(T_1) \leq m(T_2)$ . This would imply that  $m(e\Lambda') = m(T_1) = m(T_2) = 0$ , contradicting the fact that  $m(e\Lambda') + m(T_1) + m(T_2) = m(\Gamma'/\Lambda') = 1$ .  $\square$

*Proof of Theorem 1.1.* Assume that  $\Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n$ , where  $\Gamma_i = \Gamma_{i,1} *_{\Lambda_i} \Gamma_{i,2}$  is an amalgamated free product group satisfying the hypothesis of Theorem 1.1, for every  $i \in \{1, 2, \dots, n\}$ . We denote by  $G_i < \Gamma$  the product of all  $\Gamma_j$  with  $j \in \{1, 2, \dots, n\} \setminus \{i\}$ .

Let  $\Gamma \curvearrowright (X, \mu)$  be a free ergodic pmp action. Let  $A$  be a Cartan subalgebra of  $M = L^\infty(X) \rtimes \Gamma$ . For a subset  $S \subset \Gamma$ , we denote by  $e_S$  the orthogonal projection from  $L^2(M)$  onto the  $\|\cdot\|_2$  closed linear span of  $\{L^\infty(X)u_g | g \in S\}$ .

For  $i \in \{1, 2, \dots, n\}$ , we decompose  $M = (L^\infty(X) \rtimes G_i) \rtimes \Gamma_i$ . By applying Theorem 7.1 we deduce that  $A \prec_M L^\infty(X) \rtimes G_i$ . Since  $A$  is a Cartan subalgebra it follows that for every  $\varepsilon > 0$  we can find a finite set  $S \subset \Gamma$  such that  $\|x - e_{S G_i S}(x)\|_2 \leq \varepsilon$ , for all  $x \in (A)_1$ .

Thus, we can find finite sets  $S_1, S_2, \dots, S_n \subset \Gamma$  such that

$$\|x - e_{S_i G_i S_i}(x)\|_2 \leq \frac{1}{n+1}, \text{ for all } x \in (A)_1 \text{ and every } i \in \{1, 2, \dots, n\}.$$

Let  $S = \cap_{i=1}^n S_i G_i S_i$ . Then  $S$  is a finite subset of  $\Gamma$  and  $\|x - e_S(x)\|_2 \leq \frac{n}{n+1}$ , for every  $x \in (A)_1$ . Thus,  $\|e_S(u)\|_2 \geq \frac{1}{n+1}$ , for every  $u \in \mathcal{U}(A)$ . Since  $\|e_S(u)\|_2^2 = \sum_{g \in S} \|E_{L^\infty(X)}(u u_g^*)\|_2^2$ , Theorem

2.1 gives that  $A \prec_M L^\infty(X)$ . Since  $A$  and  $L^\infty(X)$  are Cartan subalgebras, [Po01, Theorem A.1] implies that they are unitarily conjugate.  $\square$

**7.2. Applications to  $W^*$ -superrigidity.** Next, we combine Theorem 1.1 with S. Popa's cocycle superrigidity [Po06a] to provide a new class of  $W^*$ -superrigid actions. In particular, we will deduce Corollary 1.2.

A free ergodic pmp action  $\Gamma \curvearrowright (X, \mu)$  is called  *$W^*$ -superrigid* if whenever  $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ , for a free ergodic pmp action  $\Lambda \curvearrowright (Y, \nu)$ , the groups  $\Gamma$  and  $\Lambda$  are isomorphic and their actions are conjugate. This means that we can find a group isomorphism  $\delta : \Gamma \rightarrow \Lambda$  and a measure space isomorphism  $\theta : X \rightarrow Y$  such that  $\theta(g \cdot x) = \delta(g) \cdot \theta(x)$ , for all  $g \in \Gamma$  and  $\mu$ -almost every  $x \in X$ .

Recall that any orthogonal representation  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H}_{\mathbb{R}})$  onto a real Hilbert space  $\mathcal{H}_{\mathbb{R}}$  gives rise to a pmp action  $\Gamma \curvearrowright (X_\pi, \mu_\pi)$ , called the *Gaussian action* associated to  $\pi$  (see for instance [Fu06, Section 2.g]).

**Theorem 7.3.** *Let  $\Gamma = \Gamma_1 *_{\Lambda} \Gamma_2$  and  $\Gamma' = \Gamma'_1 *_{\Lambda'} \Gamma'_2$  be amalgamated free product groups such that  $[\Gamma_1 : \Lambda] \geq 2$ ,  $[\Gamma_2 : \Lambda] \geq 3$ ,  $[\Gamma'_1 : \Lambda'] \geq 2$  and  $[\Gamma'_2 : \Lambda'] \geq 3$ . Suppose that there exist  $g_1, g_2, \dots, g_n \in \Gamma$  and  $g'_1, g'_2, \dots, g'_n \in \Gamma'$  such that  $\cap_{i=1}^n g_i \Lambda g_i^{-1} = \{e\}$  and  $\cap_{i=1}^n g'_i \Lambda' g'_i^{-1} = \{e\}$ .*

*Let  $G = \Gamma \times \Gamma'$  and  $\pi : G \rightarrow \mathcal{O}(\mathcal{H}_{\mathbb{R}})$  be an orthogonal representation such that*

- *the representation  $\pi|_{\Gamma}$  has stable spectral gap, i.e.  $\pi|_{\Gamma} \otimes \bar{\pi}|_{\Gamma}$  has spectral gap, and*
- *the representation  $\pi|_{\Gamma'}$  is weakly mixing, i.e.  $\pi|_{\Gamma'} \otimes \bar{\pi}|_{\Gamma'}$  has no invariant vectors.*

*Then any free ergodic pmp action  $G \curvearrowright (X, \mu)$  which can be realized as a quotient of the Gaussian action  $G \curvearrowright (X_\pi, \mu_\pi)$ , is  $W^*$ -superrigid.*

S. Popa and S. Vaes have very recently proven that the same holds when  $\Gamma$  and  $\Gamma'$  are icc weakly amenable groups that admit a proper 1-cocycle into a representation with stable spectral gap [PV11, Theorem 12.2].

*Proof.* Denote  $M = L^\infty(X) \rtimes G$  and let  $\Lambda \curvearrowright (Y, \nu)$  be a free ergodic pmp action such that we have an isomorphism  $\theta : L^\infty(Y) \rtimes \Lambda \rightarrow M$ . Then  $\theta(L^\infty(Y))$  is a Cartan subalgebra of  $M$ . Thus, by Theorem 1.1 we can find a unitary  $u \in M$  such that  $\theta(L^\infty(Y)) = uL^\infty(X)u^*$ .

This implies that the actions  $G \curvearrowright (X, \mu)$  and  $\Lambda \curvearrowright (Y, \nu)$  are orbit equivalent. Therefore, in order to show that the actions are actually conjugate, it suffices to argue that  $G \curvearrowright (X, \mu)$  is orbit equivalent superrigid.

Let us show that we can apply [Po06a, Theorem 1.3] to  $G \curvearrowright X$ . Firstly, by Corollary 6.2,  $\Gamma$  and  $\Gamma'$  have no finite normal subgroup. Thus,  $G$  has no finite normal subgroups. Secondly, by [Fu06, Theorem 1.2] the action  $G \curvearrowright X$  is s-malleable.

Thirdly, consider the unitary representation  $\rho : G \curvearrowright L^2(X_\pi) \ominus \mathbb{C}1$ . Then  $\rho$  is a subrepresentation of  $\pi \otimes \sigma$ , where  $\sigma = \oplus_{n \geq 0} \pi^{\otimes n}$ . Since  $\pi|_{\Gamma}$  has stable spectral gap and  $\pi|_{\Gamma'}$  is weakly mixing, the same properties hold for  $\rho|_{\Gamma}$  and  $\rho|_{\Gamma'}$ . Thus, the action  $\Gamma \curvearrowright X_\pi$  has stable spectral gap and the action  $\Gamma' \curvearrowright X_\pi$  is weakly mixing.

Thus, we can apply [Po06a, Theorem 1.3] to deduce that the action  $G \curvearrowright X$  is OE superrigid.  $\square$

*Proof of Corollary 1.2.* Note that the Bernoulli action  $G \curvearrowright [0, 1]^G$  can be identified with the Gaussian action associated to the left regular representation  $\lambda : G \rightarrow \mathcal{U}(\ell^2(G))$ . Since  $\Gamma$  and  $\Gamma'$  are non-amenable, the corollary follows from Theorem 7.3.

**Remark 7.4.** In [Ki10, Theorem 1.1], Y. Kida proved the following: let  $\text{Mod}^*(S)$  be the extended mapping class group of a surface of genus  $g$  with  $p$  boundary components. Suppose that  $3g + p \geq 5$  and  $(g, p) \neq (1, 2), (2, 0)$ . Let  $\Delta < \text{Mod}^*(S)$  be a finite index subgroup and  $A < \Delta$  be an infinite, almost malnormal subgroup (i.e.  $hAh^{-1} \cap A$  is finite, for all  $h \in \Delta \setminus A$ ) and denote  $\Gamma = \Delta *_A \Delta$ . Then any free ergodic pmp action  $\Gamma \curvearrowright (X, \mu)$  whose restriction to  $A$  is aperiodic is OE-superrigid.

Since  $A < \Gamma$  is weakly malnormal, Theorem 1.1 implies that all such actions of  $\Gamma$  are moreover  $W^*$ -superrigid.

**7.3. An application to  $W^*$ -rigidity.** In combination with the orbit equivalence rigidity results of N. Monod and Y. Shalom, Theorem 1.1 implies the following.

**Theorem 7.5.** *Let  $\Gamma_1, \Gamma_2, \Gamma_3$  and  $\Gamma_4$  be any non-trivial torsion-free countable groups and define  $\Gamma = (\Gamma_1 * \Gamma_2) \times (\Gamma_3 * \Gamma_4)$ . Let  $\Gamma \curvearrowright (X, \mu)$  be a free ergodic pmp action whose restrictions to  $\Gamma_1 * \Gamma_2, \Gamma_3 * \Gamma_4$  and any finite index subgroup  $\Gamma' < \Gamma$  are also ergodic.*

*Let  $\Lambda \curvearrowright (Y, \nu)$  be an arbitrary free mildly mixing pmp action.*

*If  $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ , then  $\Gamma \cong \Lambda$  and the actions  $\Gamma \curvearrowright X$  and  $\Lambda \curvearrowright Y$  are conjugate.*

Following [MS02, Definition 1.8], a measure preserving action  $\Lambda \curvearrowright (Y, \nu)$  is called *mildly mixing* if for any measurable set  $A \subset Y$  and any sequence  $\lambda_n \in \Lambda$  with  $\lambda_n \rightarrow \infty$ , one has  $\nu(\lambda_n A \Delta A) \rightarrow 0$  if and only if  $\nu(A) \in \{0, 1\}$ .

*Proof of Theorem 7.5.* By [MS02, Theorem 1.3] the groups  $\Gamma_1 * \Gamma_2$  and  $\Gamma_3 * \Gamma_4$  belong to the class  $\mathcal{C}_{\text{reg}}$ . Applying [MS02, Theorem 1.10] then gives the conclusion.  $\square$

**7.4.  $W^*$  Bass-Serre rigidity.** We next combine Theorem 1.1 with results of A. Alvarez and D. Gaboriau [AG08] to generalize part of [IPP05, Theorem 7.7] and [CH08, Theorem 6.6].

**Theorem 7.6.** *Let  $m, n \geq 2$  be integers and  $\Gamma_1, \Gamma_2, \dots, \Gamma_m, \Lambda_1, \Lambda_2, \dots, \Lambda_n$  be non-amenable groups with vanishing first  $\ell^2$ -Betti numbers. Define  $\Gamma = \Gamma_1 * \Gamma_2 * \dots * \Gamma_m$  and  $\Lambda = \Lambda_1 * \Lambda_2 * \dots * \Lambda_n$ . Let  $\Gamma \curvearrowright (X, \mu)$  and  $\Lambda \curvearrowright (Y, \nu)$  be free pmp actions such that the restrictions  $\Gamma_i \curvearrowright X$  and  $\Lambda_j \curvearrowright Y$  are ergodic, for every  $i \in \{1, 2, \dots, m\}$  and  $j \in \{1, 2, \dots, n\}$ .*

*Let  $\theta : L^\infty(X) \rtimes \Gamma \rightarrow (L^\infty(Y) \rtimes \Lambda)^t$  be an isomorphism, for some  $t > 0$ .*

*Then  $t = 1$ ,  $m = n$  and there exists a permutation  $\alpha$  of  $\{1, 2, \dots, m\}$  such that the actions  $\Gamma_i \curvearrowright X$  and  $\Lambda_{\alpha(i)} \curvearrowright Y$  are orbit equivalent, for every  $i \in \{1, 2, \dots, m\}$ .*

*Moreover, for every  $i \in \{1, 2, \dots, m\}$ , there exists a unitary element  $u_i \in L^\infty(Y) \rtimes \Lambda$  such that  $\theta(L^\infty(X)) = u_i L^\infty(Y) u_i^*$  and  $\theta(L^\infty(X) \rtimes \Gamma_i) = u_i (L^\infty(Y) \rtimes \Lambda_{\alpha(i)}) u_i^*$ .*

*Proof.* By Theorem 1.1, the  $\text{II}_1$  factor  $L^\infty(X) \rtimes \Gamma$  has a unique Cartan subalgebra, up to unitary conjugacy. Thus, we can find a unitary  $u \in (L^\infty(Y) \rtimes \Lambda)^t$  such that  $\theta(L^\infty(X)) = u(L^\infty(Y))^t u^*$ . Denoting by  $\mathcal{R}(\Gamma \curvearrowright X)$  the equivalence relation induced by the action  $\Gamma \curvearrowright X$ , it follows that  $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)^t$ . By using [Ga01] to calculate the first  $\ell^2$ -Betti number of both sides of this equation (see the end of the proof of [IPP05, Theorem 7.7]) we deduce that  $t = 1$ . Now, by [AG08, Corollary 4.20], non-amenable groups with vanishing first  $\ell^2$ -Betti number are measurably freely indecomposable. Since  $\mathcal{R}(\Gamma \curvearrowright X) = *_{i=1}^m \mathcal{R}(\Gamma_i \curvearrowright X)$  and  $\mathcal{R}(\Lambda \curvearrowright Y) = *_{j=1}^n \mathcal{R}(\Lambda_j \curvearrowright Y)$ , by applying [AG08, Theorem 5.1], the conclusion follows.  $\square$

**7.5.  $\text{II}_1$  factors with trivial fundamental group.** Theorem 1.6 also leads to a new class of groups whose actions give rise to  $\text{II}_1$  factors with trivial fundamental groups.

**Theorem 7.7.** *Let  $\Gamma_1, \Gamma_2$  be two finitely generated, countable groups with  $|\Gamma_1| \geq 2$  and  $|\Gamma_2| \geq 3$ . Denote  $\Gamma = \Gamma_1 * \Gamma_2$  and let  $\Gamma \curvearrowright (X, \mu)$  be any free ergodic pmp action.*

*Then the  $\text{II}_1$  factor  $M = L^\infty(X) \rtimes \Gamma$  has trivial fundamental group,  $\mathcal{F}(M) = \{1\}$ .*

*Proof.* By Theorem 1.6,  $L^\infty(X) \rtimes \Gamma$  has a unique Cartan subalgebra, up to unitary conjugacy. Therefore, we have that  $\mathcal{F}(M) = \mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X))$ . Since  $\beta_1^{(2)}(\Gamma) \in (0, \infty)$ , a well-known result of D. Gaboriau [Ga01] implies that  $\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X)) = \{1\}$ .  $\square$

**Remark 7.8.** Theorem 7.7 generalizes [PV08, Theorem 1.2]. Thus, it was shown in [PV08] that the conclusion of Theorem 7.7 holds, for instance, if  $\Gamma_1$  is an icc property (T) group and  $\Gamma_2$  is an infinite group. Note that Theorem 7.7 fails if the groups involved are not finitely generated. Indeed, by [PV08, Theorem 1.1] if  $\Lambda_1$  is a non-trivial group and  $\Lambda_2$  is an infinite amenable group, then  $\Gamma = \Lambda_1^{*\infty} * \Lambda_2$  does not satisfy the conclusion of Theorem 7.7. In fact, as shown in [PV08], there are free ergodic pmp actions  $\Gamma \curvearrowright X$  such that  $\mathcal{F}(L^\infty(X) \rtimes \Gamma)$  is uncountable.

**7.6. Absence of Cartan subalgebras.** Finally, Theorem 7.1 allows us to provide a new class of  $\text{II}_1$  factors without Cartan subalgebras:

**Corollary 7.9.** *Let  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$  be an amalgamated free product group such that  $[\Gamma_1 : \Lambda] \geq 2$  and  $[\Gamma_2 : \Lambda] \geq 3$ . Assume that there exist  $g_1, g_2, \dots, g_n \in \Gamma$  such that  $\cap_{i=1}^n g_i \Lambda g_i^{-1} = \{e\}$ .*

*Then  $N \bar{\otimes} L(\Gamma)$  does not have a Cartan subalgebra, for any  $\text{II}_1$  factor  $N$ .*

*Proof of Corollary 7.9.* Let  $N$  be a  $\text{II}_1$  factor and denote  $M = N \bar{\otimes} L(\Gamma)$ . Assume by contradiction that  $M$  has a Cartan subalgebra  $A$ . Since  $M = N \rtimes \Gamma$ , where  $\Gamma$  acts trivially on  $N$ , Theorem 7.1 implies  $A \prec_M N$ . By taking relative commutants (see [Va07, Lemma 3.5]) we get that  $L(\Gamma) \prec_M A' \cap M = A$ . Since  $A$  is abelian, while  $\Gamma$  is non-amenable, we derive a contradiction.  $\square$

## 8. CARTAN SUBALGEBRAS OF AFP ALGEBRAS AND CLASSIFICATION OF $\text{II}_1$ FACTORS ARISING FROM FREE PRODUCT EQUIVALENCE RELATIONS

In this section we prove Theorem 1.3 and Corollary 1.4.

**8.1. Proof of Theorem 1.3.** Let  $A$  be a Cartan subalgebra of  $M = M_1 *_B M_2$ . Recall that  $B$  is amenable,  $pM_1p \neq pBp \neq pM_2p$ , for any non-zero projection  $p \in B$ , and that either

- (1)  $M_1$  and  $M_2$  have no amenable direct summands, or
- (2)  $M$  does not have property  $\Gamma$ .

We claim that  $M \not\prec_M M_i$ , for any  $i \in \{1, 2\}$ . Assume by contradiction that  $M \prec_M M_i$ , for some  $i \in \{1, 2\}$ . By Theorem 2.1 we can find projections  $p \in M, q \in M_i$ , a non-zero partial isometry  $v \in qMp$  such that  $v^*v = p$ , and a  $*$ -homomorphism  $\phi : pMp \rightarrow qM_iq$  such that  $\phi(x)v = vx$ , for all  $x \in pMp$ . Since  $M$  is a non-amenable factor and  $B$  is amenable, we have that  $M \not\prec_M B$ . Thus, by [Va07, Remark 3.8] we can moreover assume that  $\phi(pMp) \not\prec_{M_i} B$ .

Then [IPP05, Theorem 1.1] implies that  $\phi(pMp)' \cap qM_iq \subset qM_iq$ . In particular,  $q_0 := vv^* \in qM_iq$ . From this we get that  $q_0 M q_0 = q_0 M_i q_0$ . Let  $j \in \{1, 2\} \setminus \{i\}$  and  $x \in M_j \ominus B$ . Then the orthogonal projection of  $q_0 x q_0$  onto  $(L^2(M_i) \ominus L^2(B)) \otimes_B (L^2(M_j) \ominus L^2(B)) \otimes_B (L^2(M_i) \ominus L^2(B))$  is equal to  $(q_0 - E_B(q_0))x(q_0 - E_B(q_0))$ . Since  $q_0 x q_0 \in M_i$ , we deduce that  $q_0 - E_B(q_0) = 0$ . Thus,  $q_0 \in B$  and  $q_0 M_j q_0 \subset q_0 M_i q_0 \cap q_0 M_j q_0 = q_0 B q_0$ . This contradicts our assumption that  $q_0 M_j q_0 \neq q_0 B q_0$ .

Next, consider  $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{F}_2))$  and the free malleable deformation  $\{\theta_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\tilde{M})$ . Let  $N = \{u_g M u_g^* | g \in \mathbb{F}_2\}''$ . Since  $\tilde{M} = N \rtimes \mathbb{F}_2$ , by applying Theorem 2.8 we have two cases:

**Case a.**  $\theta_t(A) \prec_{\tilde{M}} N$ , for some  $t \in (0, 1)$ .

**Case b.**  $\theta_t(M)$  is amenable relative to  $N$  inside  $\tilde{M}$ , for any  $t \in (0, 1)$ .

In **Case a**, Theorem 3.2 gives that either  $A \prec_M B$  or  $M \prec_M M_i$ , for some  $i \in \{1, 2\}$ . Since the latter is impossible by the above, the conclusion holds in this case.

To finish the proof it is enough to argue that **Case b** contradicts each of the above assumptions (1) and (2). Indeed, by applying Theorem 4.1 we get that  $M_i p_i$  is amenable relative to  $B$ , for some non-zero projection  $p_i \in \mathcal{Z}(M_i)$  and some  $i \in \{1, 2\}$ . Since  $B$  is amenable, this would imply that either  $M_1$  or  $M_2$  has an amenable direct summand, contradicting assumption (1).

Also, by applying Theorem 5.1 we would get that either  $M$  has property  $\Gamma$ ,  $M \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or  $M$  is amenable relative to  $B$  (hence  $M$  is amenable and therefore isomorphic to the hyperfinite  $\text{II}_1$  factor). Since the hyperfinite  $\text{II}_1$  factor has property  $\Gamma$ , this contradicts assumption (2).

**Remark 8.1.** Theorem 1.3 requires that  $M = M_1 *_B M_2$  is a factor. Note that when  $B$  is a type I von Neumann algebra, [HV12, Theorem 5.8] and [Ue12, Theorem 4.3] provide general conditions which guarantee that  $M$  is a factor.

**8.2. Proof of Corollary 1.4.** Denote  $M = L(\mathcal{R})$ ,  $M_1 = L(\mathcal{R}_1)$ ,  $M_2 = L(\mathcal{R}_2)$  and  $B = L^\infty(X)$ . Then  $M = M_1 *_B M_2$ . Since the restrictions of  $\mathcal{R}_1$  and  $\mathcal{R}_2$  to any set of positive measure have infinite orbits, we get that  $p M_1 p \neq p B p \neq p M_2 p$ , for any non-zero projection  $p \in B$ .

Now, if the restrictions of  $\mathcal{R}_1$  and  $\mathcal{R}_2$  to any set of positive measure are non-hyperfinite, then  $M_1$  and  $M_2$  have no amenable direct summand [CFW81].

Next, let us show that if  $\mathcal{R}$  is strongly ergodic, then  $M$  does not have property  $\Gamma$ . Since the restrictions of  $\mathcal{R}_1$  and  $\mathcal{R}_2$  to any set of positive measure have infinite orbits, [IKT08, Lemma 2.6] provides  $\theta_1 \in [\mathcal{R}_1]$  and  $\theta_2, \theta_3 \in [\mathcal{R}_2]$  such that  $\theta_1(x) \neq x, \theta_2(x) \neq x, \theta_3(x) \neq x$  and  $\theta_2(x) \neq \theta_3(x)$ , for  $\mu$ -almost every  $x \in X$ . Thus the unitaries  $u = u_{\theta_1} \in M_1$ ,  $v = u_{\theta_2} \in M_2$  and  $w = u_{\theta_3} \in M_2$  satisfy  $E_B(u) = E_B(v) = E_B(w) = E_B(w^*v) = 0$ . By Lemma 6.1 we get that  $M' \cap M^\omega \subset B^\omega$ .

Since  $\mathcal{R}$  is strongly ergodic, we have that  $M' \cap B^\omega = \mathbb{C}$ , which shows that  $M$  does not have property  $\Gamma$ .

Altogether by applying Theorem 1.3 we deduce that if  $A$  is a Cartan subalgebra of  $M$ , then  $A \prec_M B$ . Hence, by [Po01, Theorem A.1] it follows that  $A$  and  $B$  are unitarily conjugate.

Finally, let  $\mathcal{S}$  be a countable measure preserving equivalence relation on a probability space  $(Z, \nu)$  and  $\theta : L(\mathcal{S}) \rightarrow M$  be an isomorphism. Then  $\theta(L^\infty(Z))$  is a Cartan subalgebra of  $M$  and so it must be conjugate to  $B$ . This shows that the inclusions  $L^\infty(X) \subset L(\mathcal{R})$  and  $L^\infty(Z) \subset L(\mathcal{S})$  are isomorphic, hence  $\mathcal{R} \cong \mathcal{S}$ .  $\square$

**Remark 8.2.** This proof moreover shows that if  $v \in H^2(\mathcal{R}, \mathbb{T})$  is any 2-cocycle, then  $L^\infty(X)$  is the unique Cartan subalgebra of the  $\text{II}_1$  factor  $L(\mathcal{R}, v)$ , up to unitary conjugacy. Thus, if  $L(\mathcal{R}, w) \cong L(\mathcal{S}, v)$ , for any ergodic countable measure preserving equivalence relation  $\mathcal{S}$  on a standard probability space  $(Y, \nu)$  and any 2-cocycle  $w \in H^2(\mathcal{S}, \mathbb{T})$ , then  $\mathcal{R} \cong \mathcal{S}$  and the cocycles  $v$  and  $w$  are cohomologous. More precisely, there exists an isomorphism of probability spaces  $\theta : X \rightarrow Y$  such that  $(\theta \times \theta)(\mathcal{R}) = \mathcal{S}$  and  $[v \circ (\theta \times \theta \times \theta)] = [w]$  in  $H^2(\mathcal{R}, \mathbb{T})$  (see [FM77]).

## 9. NORMALIZERS OF AMENABLE SUBALGEBRAS OF AFP ALGEBRAS

In the first part of this section we prove Theorem 1.6 and Corollary 1.7, and then deduce Corollary 1.5.

**9.1. Proof of Theorem 1.6.** Let  $A \subset M = M_1 *_B M_2$  be a von Neumann subalgebra that is amenable relative to  $B$ . Suppose that  $P = \mathcal{N}_M(A)''$  satisfies  $P' \cap M^\omega = \mathbb{C}1$ .

Let  $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{F}_2))$  and  $\{\theta_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\tilde{M})$  the associated free malleable deformation. Let  $N = \{u_g M u_g^* | g \in \mathbb{F}_2\}''$  and recall that  $\tilde{M} = N \rtimes \mathbb{F}_2$ . Since  $A$  is amenable relative to  $B$  and  $\theta_t(B) = B \subset N$ , we deduce that  $\theta_t(A)$  is amenable relative to  $N$ , for any  $t \in \mathbb{R}$ .

By Theorem 2.8 either there exists  $t \in (0, 1)$  such that  $\theta_t(A) \prec_{\tilde{M}} N$  or else  $\theta_t(P)$  is amenable relative to  $N$  inside  $\tilde{M}$ , for every  $t \in (0, 1)$ .

In the first case, Theorem 3.2 gives that either  $A \prec_M B$  or  $P \prec_M M_i$ , for some  $i \in \{1, 2\}$ . In the second case, Theorem 5.1 implies that either  $P \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or  $P$  is amenable relative to  $B$  inside  $M$ . Altogether, the conclusion follows.  $\square$

**9.2. Proof of Corollary 1.7.** We establish the following more precise version of Corollary 1.7. If  $P \subset pMp$  and  $Q \subset M$  are von Neumann subalgebras then we write  $P \prec_M^s Q$  if  $Pp' \prec_M Q$ , for any non-zero projection  $p' \in P' \cap pMp$ .

**Corollary 9.1.** *Let  $(M_1, \tau_1), (M_2, \tau_2)$  be two tracial von Neumann algebras. Let  $M = M_1 * M_2$  and  $A \subset M$  be a diffuse amenable von Neumann subalgebra. Denote  $P = \mathcal{N}_M(A)''$ .*

*Then we can find projections  $p_1, p_2, p_3 \in \mathcal{Z}(P)$  satisfying  $p_1 + p_2 + p_3 = 1$  and*

- (1)  $Pp_1 \prec_M^s M_1$ ,
- (2)  $Pp_2 \prec_M^s M_2$ , and
- (3)  $Pp_3$  is amenable.

*Moreover, if  $M_1$  and  $M_2$  are factors, then we can find unitary elements  $u_1, u_2 \in M$  such that  $u_1 P p_1 u_1^* \subset M_1$  and  $u_2 P p_2 u_2^* \subset M_2$ .*

*Proof.* If a non-zero projection  $p \in \mathcal{Z}(P) = P' \cap M$  satisfies  $Pp \prec_M M_i$ , for some  $i \in \{1, 2\}$ , then there exists a non-zero projection  $p' \in \mathcal{Z}(P)p$  such that  $Pp' \prec_M^s M_i$ . Thus, in order to get the first part of the conclusion, it suffices to argue that if  $p \in \mathcal{Z}(P)$  is a non-zero projection such that  $Pp$  has no amenable direct summand, then either  $Pp \prec_M M_1$  or  $Pp \prec_M M_2$ .

By Theorem 2.7 we can find projections  $e, f \in \mathcal{Z}((Pp)' \cap pMp) \cap \mathcal{Z}((Pp)' \cap (pMp)^\omega)$  such that

- $e + f = p$ .
- $((Pp)' \cap (pMp)^\omega)e$  is completely atomic and  $((Pp)' \cap (pMp)^\omega)e = ((Pp)' \cap (pMp))e$ .
- $((Pp)' \cap (pMp)^\omega)f$  is diffuse.

Since  $p \neq 0$ , we have that either  $e \neq 0$  or  $f \neq 0$ .

In the first case, let  $e_0 \in ((Pp)' \cap (pMp)^\omega)e$  be a minimal non-zero projection. Then we have that  $e_0 \in p(P' \cap M^\omega)p \cap p(P' \cap M)p$  and  $e_0(P' \cap M^\omega)e_0 = \mathbb{C}e_0$ . Therefore,  $Pe_0$  is a von Neumann subalgebra of  $e_0 M e_0$  such that  $(Pe_0)' \cap (e_0 M e_0)^\omega = \mathbb{C}e_0$ .

Note that  $Pe_0 \subset \mathcal{N}_{e_0 M e_0}(Ae_0)''$ . Also, we have that  $A$  and hence  $Ae_0$  is diffuse. By applying Theorem 1.6 (in the case  $B = \mathbb{C}$ ) we deduce that either  $Pe_0 \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or  $Pe_0$  is amenable. Since  $e_0 \leq p$ ,  $Pe_0$  cannot be amenable. Thus, we must have that  $Pe_0 \prec_M M_i$  and hence that  $Pp \prec_M M_i$ , for some  $i \in \{1, 2\}$ .

In the second case, we have that  $f \in p(P' \cap M^\omega)p \cap p(P' \cap M)p$  and that  $f(P' \cap M^\omega)f$  is diffuse. Thus,  $Pf$  is a von Neumann subalgebra of  $fMf$  such that  $(Pf)' \cap (fMf)^\omega$  is diffuse.

By applying Theorem 6.3 (with  $B = \mathbb{C}$ ) we deduce that either  $Pf \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or  $Pf_0$  is amenable, for some non-zero projection  $f_0 \in \mathcal{Z}((Pf)' \cap fMf)$ . Since  $f_0 \leq p$ , the latter is impossible. Thus we conclude that  $Pp \prec_M M_i$ , for some  $i \in \{1, 2\}$ , in this case as well.

The moreover part now follows by repeating the proof of [IPP05, Theorem 5.1 (2)].  $\square$

**9.3. Proof of Corollary 1.5.** Assume by contradiction that  $M = M_1 * M_2$  has a Cartan subalgebra  $A$ . Since  $M_1 \neq \mathbb{C} \neq M_2$  and  $\dim(M_1) + \dim(M_2) \geq 5$ , by [Ue10, Theorem 4.1] there exists a non-zero central projection  $z \in M$  such that  $Mz$  is a  $\text{II}_1$  factor without property  $\Gamma$ , while  $M(1 - z)$  is completely atomic. In particular,  $M$  is not amenable.

To derive a contradiction we treat separately two cases

**Case 1.**  $M_1$  and  $M_2$  are completely atomic.

**Case 2.** Either  $M_1$  or  $M_2$  has a diffuse direct summand.

In the first case, since  $\mathcal{N}_M(A)'' = M$ , Corollary 9.1 yields projections  $p_1, p_2, p_3 \in \mathcal{Z}(M)$  such that  $p_1 + p_2 + p_3 = 1$ ,  $Mp_1 \prec_M^s M_1$ ,  $Mp_2 \prec_M^s M_2$  and  $Mp_3$  is amenable. Since  $M_1, M_2$  are completely atomic, it follows that  $Mp_1, Mp_2$  are completely atomic. Altogether, we derive that  $M$  is amenable, a contradiction.

In the second case, we may assume for instance that  $M_1$  has a diffuse direct summand. Hence, there exists a non-zero projection  $p \in \mathcal{Z}(M_1)$  such that  $M_1p$  is diffuse. Since  $M(1 - z)$  is completely atomic, we must have that  $p \leq z$ .

Define  $N = (\mathbb{C}p + M_1(1 - p)) \vee M_2$ . Then by [Ue10, Lemma 2.2] we have that  $M_1p$  and  $pNp$  are free and together generate  $pMp$ , i.e.  $pMp = M_1p * pNp$ . We also have that  $pNp \neq \mathbb{C}p$ . Indeed, since  $M_2 \neq \mathbb{C}$ , there exists a projection  $q \in M_2$  with  $q \neq 0, 1$ . Then  $pqp \in pNp$  and  $pqp = \tau(q)p + p(q - \tau(q))p$ . This clearly implies that  $pqp \notin \mathbb{C}p$ .

Now, note that  $Az$  is a Cartan subalgebra of  $Mz$ . Since  $Mz$  is a factor and  $p \in Mz$ , it follows that  $pMp$  also has a Cartan subalgebra. Since  $Mz$  does not have property  $\Gamma$ , it follows that  $pMp$  does not have property  $\Gamma$  as well. On the other hand, since  $pMp = M_1p * pNp$  and  $M_1p \neq \mathbb{C}p \neq pNp$ , by applying Theorem 1.3 (2) in the case  $B = \mathbb{C}p$ , we conclude that  $pMp$  does not have a Cartan subalgebra. This leads to the desired contradiction.  $\square$

**9.4. Strongly solid von Neumann algebras.** Our final aim is to prove Theorem 1.8. We begin by introducing some terminology motivated by the proof of [Po03, Theorem 3.1].

**Definition 9.2.** [Po03] Let  $(M, \tau)$  be a tracial von Neumann algebra and  $B \subset M$  be a von Neumann subalgebra. We say that the inclusion  $B \subset M$  is *mixing* if for every  $x, y \in M \ominus B$  and any sequence  $b_n \in (B)_1$  such that  $b_n \rightarrow 0$  weakly we have that  $\|E_B(xb_ny)\|_2 \rightarrow 0$ .

This notion has been considered in [JS06] and [CJM10], where several examples of mixing inclusions of von Neumann algebras were exhibited.

**Remark 9.3.** Let  $B \subset M$  be tracial von Neumann algebras.

- (1) It is easy to see that the inclusion  $B \subset M$  is mixing if and only if the  $B$ - $B$  bimodule  $L^2(M) \ominus L^2(B)$  is mixing in the sense of [PS09, Definition 2.3].
- (2) In particular, the inclusion  $B \subset M$  is mixing whenever the  $B$ - $B$  bimodule  $L^2(M) \ominus L^2(B)$  is isomorphic to a sub-bimodule of  $\bigoplus_{i=1}^\infty (L^2(B) \otimes L^2(B))$ . This is the case, for instance, if we can decompose  $M = B * C$ , for some von Neumann subalgebra  $C \subset M$  (see the proof of [Po06b, Lemma 2.2]).



- (3) Let  $\Lambda < \Gamma$  be an inclusion of countable groups. Then the inclusion of group von Neumann algebras  $L(\Lambda) \subset L(\Gamma)$  is mixing if and only if  $g\Lambda g^{-1} \cap \Lambda$  is finite, for every  $g \in \Gamma \setminus \Lambda$  (see [JS06, Theorem 3.5] and the proof of Corollary 9.8).
- (4) Let  $(D, \tau)$  be a tracial von Neumann algebra and  $\Gamma \curvearrowright D$  be a mixing trace preserving action. Then the inclusion  $L(\Gamma) \subset D \rtimes \Gamma$  is mixing (see the proof of [Po03, Lemma 3.4]).

In order to prove Theorem 1.8 we need two technical lemmas.

**Lemma 9.4.** [Po03] *Let  $(M, \tau)$  be a tracial von Neumann algebra and  $B \subset M$  be a von Neumann subalgebra. Assume that the inclusion  $B \subset M$  is mixing. Let  $A \subset pMp$  be a diffuse von Neumann subalgebra, for some projection  $p \in M$ , and denote  $P = \mathcal{N}_{pMp}(A)''$ . Then we have*

- (1) *If  $A \subset B$ , then  $P \subset B$ .*
- (2) *If  $A \prec_M B$ , then  $P \prec_M B$ .*

*Proof.* For the reader's convenience let us briefly indicate how the lemma follows from [Po03].

Recall that the *quasi-normalizer* of a von Neumann subalgebra  $Q \subset M$ , denoted  $q\mathcal{N}_M(Q)$ , consists of those elements  $x \in M$  for which we can find  $x_1, \dots, x_n \in M$  such that  $xQ \subset \sum_{i=1}^n Qx_i$  and  $Qx \subset \sum_{i=1}^n x_iQ$  (see [Po01, Section 1.4.2]). Note that  $\mathcal{N}_M(Q) \subset q\mathcal{N}_M(Q)$ .

Let  $Q \subset rBr$  be a diffuse von Neumann subalgebra, for some projection  $r \in B$ . Since the inclusion  $B \subset M$  is mixing, the proof of [Po03, Theorem 3.1] shows that the quasi-normalizer of  $Q$  in  $rMr$  is contained in  $rBr$  (see also the proof of [IPP05, Theorem 1.1]). This fact implies (1).

To prove (2), assume that  $A \prec_M B$ . Then we can find projections  $q \in A$ ,  $r \in B$ , a non-zero partial isometry  $v \in rMq$  and a  $*$ -homomorphism  $\phi : qAq \rightarrow rBr$  such that  $\phi(x)v = vx$ , for all  $x \in qAq$ . Since  $\phi(qAq) \subset rBr$  is diffuse, the previous paragraph gives that  $q\mathcal{N}_{rMr}(\phi(qAq)) \subset rBr$ .

Next, let  $u \in \mathcal{N}_{pMp}(A)$ . Following the proof of [Po03, Lemma 3.5], let  $z \in A$  be a central projection such that  $z = \sum_{j=1}^m v_j v_j^*$ , for some partial isometries  $\{v_j\}_{j=1}^m$  in  $A$  satisfying  $v_j^* v_j \leq q$ . We claim that  $qzuqz \in qMq$  belongs to the quasi-normalizer of  $qAq$ . Indeed, we have

$$qzuqz(qAq) \subset qzuA = qzAu = qAzu \subset \sum_{j=1}^m (qAv_j)v_j^*u \subset \sum_{j=1}^m (qAq)v_j^*u$$

and similarly  $(qAq)qzuqz \subset \sum_{j=1}^m uv_j(qAq)$ .

Now, it is clear that if  $x \in q\mathcal{N}_{qMq}(qAq)$ , then  $vxv^* \in q\mathcal{N}_{rMr}(\phi(qAq))$ . By combining the last two paragraphs we derive that  $vqzuqzv^* \in rBr$ . Since the central projections  $z$  of the desired form approximate arbitrarily well the central support of  $q$ , we deduce that  $vquqv^* \in rBr$ . Thus,  $vuv^* \in rBr$ , for all  $u \in \mathcal{N}_{pMp}(A)$ . Hence  $vPv^* \subset rBr$  and so we conclude that  $P \prec_M B$ .  $\square$

**Lemma 9.5.** *Let  $(M, \tau)$  be a tracial von Neumann algebra and  $B \subset M$  be a von Neumann subalgebra. Assume that the inclusion  $B \subset M$  is mixing.*

*Let  $P \subset pMp$  be a separable von Neumann subalgebra, for some projection  $p \in M$ , and  $\omega$  be a free ultrafilter on  $\mathbb{N}$ . Assume that  $P' \cap (pMp)^\omega$  is diffuse and  $P' \cap (pMp)^\omega \prec_{M^\omega} B^\omega$ .*

*Then  $P \prec_M B$ .*

*Proof.* We first prove the conclusion under the additional assumption that  $P' \cap pMp = \mathbb{C}p$ . We assume for simplicity that  $p = 1$ , the general case being treated similarly. Denote  $P_\omega = P' \cap M^\omega$  and let  $\{y_n\}_{n \geq 1}$  be a  $\|\cdot\|_2$  dense sequence in  $(P)_1$ .

Since  $P_\omega \prec_{M^\omega} B^\omega$ , we can find  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in M^\omega$  and  $\delta > 0$  such that

$$(9.1) \quad \sum_{i=1}^n \|E_{B^\omega}(a_i u b_i)\|_2^2 > \delta, \quad \text{for all } u \in \mathcal{U}(P_\omega).$$

For every  $i \in \{1, 2, \dots, n\}$ , write  $a_i = (a_{i,k})_k$  and  $b_i = (b_{i,k})_k$ , for some  $a_{i,k}, b_{i,k} \in M$ .

**Claim 1.** There exists  $k \in \mathbb{N}$  such that

$$(9.2) \quad \sum_{i=1}^n \|E_{B^\omega}(a_{i,k} u b_{i,k})\|_2^2 \geq \delta, \quad \text{for all } u \in \mathcal{U}(P_\omega).$$

*Proof of Claim 1.* Suppose that the claim is false and fix  $k \in \mathbb{N}$ . Then there is a unitary  $u_k \in P_\omega$  such that  $\sum_{i=1}^n \|E_{B^\omega}(a_{i,k} u_k b_{i,k})\|_2^2 < \delta$ . Write  $u_k = (u_{k,l})_l$ , where  $u_{k,l} \in \mathcal{U}(M)$ . Then the last inequality rewrites as  $\lim_{l \rightarrow \omega} \sum_{i=1}^n \|E_B(a_{i,k} u_{k,l} b_{i,k})\|_2^2 < \delta$ . Also, we have that  $\lim_{l \rightarrow \omega} \|[u_{k,l}, y_j]\|_2 = \|[u_k, y_j]\|_2 = 0$ , for all  $j \geq 1$ . It altogether follows that we can find  $l \in \mathbb{N}$  such that  $U_k := u_{k,l}$  satisfies  $\sum_{i=1}^n \|E_B(a_{i,k} U_k b_{i,k})\|_2^2 < \delta$  and  $\sum_{j=1}^k \|[U_k, y_j]\|_2 \leq \frac{1}{k}$ .

It is then clear that the unitary  $U = (U_k)_k$  belongs to  $P_\omega$  and satisfies  $\sum_{i=1}^n \|E_{B^\omega}(a_i U b_i)\|_2^2 \leq \delta$ . This contradicts inequality 9.1.  $\square$

We next use an idea of S. Vaes (see the proof of [Io11a, Theorem 3.1]).

Denote by  $\mathcal{K}$  the  $\|\cdot\|_2$  closure of the linear span of the set  $\{axb | a, b \in M, x \in B^\omega \ominus B\}$ . Then  $\mathcal{K}$  is a Hilbert subspace of  $L^2(M^\omega)$  that is an  $M$ - $M$  bimodule. Denote by  $e$  the orthogonal projection from  $L^2(M^\omega)$  onto  $\mathcal{K}$ .

Since  $P_\omega$  is diffuse we can find a unitary  $u \in P_\omega$  such that  $\tau(u) = 0$ . Since  $E_M(u) \in P' \cap M$  and  $P' \cap M = \mathbb{C}1$ , it follows that  $E_M(u) = \tau(E_M(u))1 = 0$ .

Let  $\xi = e(u)$ . We claim that  $\xi \neq 0$ . Let  $k \in \mathbb{N}$  as in Claim 1 and  $\eta = \sum_{i=1}^n a_{i,k}^* E_{B^\omega}(a_{i,k} u b_{i,k}) b_{i,k}^*$ . Note that  $E_B(E_{B^\omega}(a_{i,k} u b_{i,k})) = E_B(a_{i,k} u b_{i,k}) = E_B(E_M(a_{i,k} u b_{i,k})) = E_B(a_{i,k} E_M(u) b_{i,k}) = 0$ . Thus  $E_{B^\omega}(a_{i,k} u b_{i,k}) \in B^\omega \ominus B$ , for all  $i \in \{1, 2, \dots, n\}$ , hence  $\eta \in \mathcal{K}$ . On the other hand, inequality 9.2 rewrites as  $\langle u, \eta \rangle \geq \delta$ . Combining the last two facts gives that  $\xi \neq 0$ .

Since  $\mathcal{K}$  is an  $M$ - $M$  bimodule and  $u$  commutes with  $P$  it follows that  $y\xi = \xi y$ , for all  $y \in P$ . Thus  $\langle y\xi y^*, \xi \rangle = \|\xi\|_2^2 > 0$ , for all  $y \in \mathcal{U}(P)$ . To finish the proof we use a second claim.

**Claim 2.** Let  $v_n, w_n \in (M)_1$  be two sequences such that  $\|E_B(a_2^* v_n a_1)\|_2 \rightarrow 0$ , for all  $a_1, a_2 \in M$ . Then for all  $\xi_1, \xi_2 \in \mathcal{K}$  we have that  $\langle v_n \xi_1 w_n, \xi_2 \rangle \rightarrow 0$ , as  $n \rightarrow \infty$ .

*Proof of Claim 2.* It suffices to prove the conclusion for  $\xi_1$  and  $\xi_2$  of the form  $\xi_1 = a_1 x_1 b_1$  and  $\xi_2 = a_2 x_2 b_2$ , for some  $a_1, a_2, b_1, b_2 \in M$  and  $x_1, x_2 \in (B^\omega \ominus B)_1$ . In this case, we have

$$|\langle v_n \xi_1 w_n, \xi_2 \rangle| = |\tau(x_2^* a_2^* v_n a_1 x_1 b_1 w_n b_2^*)| \leq \|E_{B^\omega}(a_2^* v_n a_1 x_1 b_1 w_n b_2^*)\|_2.$$

Since the inclusion  $B \subset M$  is mixing, we have  $E_{B^\omega}(cxd) = 0$ , for all  $c, d \in M \ominus B$  and  $x \in B^\omega \ominus B$ . Thus  $E_{B^\omega}(a_2^* v_n a_1 x_1 b_1 w_n b_2^*) = E_B(a_2^* v_n a_1) x_1 E_B(b_1 w_n b_2^*)$ . In combination with the last inequality this implies that  $|\langle v_n \xi_1 w_n, \xi_2 \rangle| \leq \|E_B(a_2^* v_n a_1)\|_2 \rightarrow 0$ .  $\square$

Now, if the conclusion  $P \prec_M B$  is false, then we can find a sequence of unitary elements  $y_n \in P$  such that  $\|E_B(a_2^* y_n a_1)\|_2 \rightarrow 0$ , for all  $a_1, a_2 \in M$ . Claim 2 then implies that  $\langle y_n \xi y_n^*, \xi \rangle \rightarrow 0$ , contradicting the fact that  $\langle y_n \xi y_n^*, \xi \rangle = \|\xi\|_2^2 > 0$ , for all  $n$ . This finishes the proof of Lemma 9.5 under the additional assumption that  $P' \cap pMp = \mathbb{C}p$ .

In general, assume again for simplicity that  $p = 1$ . Then we can find projections  $\{p_n\}_{n \geq 0} \in P' \cap M$  such that  $p_0 \in \mathcal{Z}(P' \cap M)$  and  $(P' \cap M)p_0$  is diffuse,  $p_n \in P' \cap M$  is a minimal projection, for all

$n \geq 1$ , and  $\sum_{n \geq 0} p_n = 1$ . Since  $P_\omega \prec_{M^\omega} B^\omega$  we can find  $n$  such that  $p_n \neq 0$  and  $p_n P_\omega p_n \prec_{M^\omega} B^\omega$ . To derive the conclusion, we treat separately two cases.

Firstly, assume that  $n = 0$ . Since  $((Pp_0)' \cap p_0 M p_0)^\omega \subset (Pp_0)' \cap (p_0 M p_0)^\omega = p_0 P_\omega p_0$  and  $p_0 P_\omega p_0 \prec_{M^\omega} B^\omega$ , it easily follows that  $(Pp_0)' \cap p_0 M p_0 \prec_M B$ . Since  $(Pp_0)' \cap p_0 M p_0 = (P' \cap M)p_0$  is diffuse, Lemma 9.4 readily gives that  $Pp_0 \prec_M B$  and hence  $P \prec_M B$ .

Secondly, suppose that  $n \geq 1$ . Since  $p_n \in P' \cap M$  is a minimal projection we get that  $(Pp_n)' \cap p_n M p_n = \mathbb{C}p_n$ . Also, we have that  $(Pp_n)' \cap (p_n M p_n)^\omega = p_n P_\omega p_n$  is diffuse and satisfies  $(Pp_n)' \cap (p_n M p_n)^\omega \prec_{M^\omega} B^\omega$ . By applying the first part of the proof to the subalgebra  $Pp_n \subset p_n M p_n$  we deduce that  $Pp_n \prec_M B$  and hence that  $P \prec_M B$ .  $\square$

*Proof of Theorem 1.8.* Since the inclusions  $B \subset M_1, B \subset M_2$  are mixing, it follows easily that the inclusion  $B \subset M$  is mixing. We claim that the inclusion  $M_i \subset M$  is also mixing, for  $i \in \{1, 2\}$ .

To this end, let  $j \in \{1, 2\}$  with  $j \neq i$ . Let  $b_n \in (M_i)_1$  be a sequence such that  $b_n \rightarrow 0$  weakly. The claim is equivalent to showing that  $\|E_{M_i}(x^* b_n y)\|_2 \rightarrow 0$ , for all  $x, y \in M \ominus M_i$ . We may assume that  $x, y$  are of the following form:  $x = x_1 x_2 \dots x_m$  and  $y = y_1 y_2 \dots y_n$ , where  $x_1 \in M_i, x_2 \in M_j \ominus B, x_3 \in M_i \ominus B \dots$  and  $y_1 \in M_i, y_2 \in M_j \ominus B, y_3 \in M_i \ominus B \dots$ , for some integers  $m, n \geq 2$ . We may also assume that  $\|x_k\| \leq 1$  and  $\|y_l\| \leq 1$ , for all  $1 \leq k \leq m$  and  $1 \leq l \leq n$ .

A simple computation shows that  $E_{M_i}(x^* b_n y) = E_{M_i}(x_m^* \dots x_3^* E_B(x_2^* E_B(x_1^* b_n y_1) y_2) y_3 \dots y_n)$ . Thus, we get that  $\|E_{M_i}(x^* b_n y)\|_2 \leq \|E_B(x_2^* E_B(x_1^* b_n y_1) y_2)\|_2$ . Since  $b_n \rightarrow 0$  weakly, we have that  $E_B(x_1^* b_n y_1) \rightarrow 0$  weakly. Since  $x_2, y_2 \in M_j \ominus B$  and the inclusion  $B \subset M_j$  is mixing, it follows that  $\|E_B(x_2^* E_B(x_1^* b_n y_1) y_2)\|_2 \rightarrow 0$ . This proves that  $\|E_{M_i}(x^* b_n y)\|_2 \rightarrow 0$  and implies the claim.

Now, to show that  $M$  is strongly solid, fix a diffuse amenable von Neumann subalgebra  $A \subset M$  and denote  $P = \mathcal{N}_M(A)''$ . Suppose by contradiction that  $P$  is not amenable and let  $z \in \mathcal{Z}(P)$  be the largest projection such that  $Pz$  is amenable. Then  $p = 1 - z \neq 0$ .

By Theorem 2.7 we can find projections  $e, f \in \mathcal{Z}((Pp)' \cap pMp) \cap \mathcal{Z}((Pp)' \cap (pMp)^\omega)$  such that

- $e + f = p$ .
- $((Pp)' \cap (pMp)^\omega)e$  is completely atomic and  $((Pp)' \cap (pMp)^\omega)e = ((Pp)' \cap (pMp))e$ .
- $((Pp)' \cap (pMp)^\omega)f$  is diffuse.

Since  $p \neq 0$ , we have that either  $e \neq 0$  or  $f \neq 0$ .

In the first case, let  $e_0 \in ((Pp)' \cap (pMp)^\omega)e$  be a minimal non-zero projection. Then we have that  $e_0 \in p(P' \cap M^\omega)p \cap p(P' \cap M)p$  and  $e_0(P' \cap M^\omega)e_0 = \mathbb{C}e_0$ . Therefore,  $Pe_0$  is a von Neumann subalgebra of  $e_0 M e_0$  such that  $(Pe_0)' \cap (e_0 M e_0)^\omega = \mathbb{C}e_0$ . Note that  $Pe_0 \subset \mathcal{N}_{e_0 M e_0}(Ae_0)''$ . Theorem 1.6 implies that either  $Ae_0 \prec_M B$ ,  $Pe_0 \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or  $Pe_0$  is amenable relative to  $B$ . Moreover if,  $Ae_0 \prec_M B$ , then since the inclusion  $B \subset M$  is mixing, Lemma 9.4 gives that  $Pe_0 \prec_M B$ .

In the second case, we have that  $f \in p(P' \cap M^\omega)p \cap p(P' \cap M)p$  and that  $f(P' \cap M^\omega)f$  is diffuse. Thus,  $Pf$  is a von Neumann subalgebra of  $fMf$  such that  $(Pf)' \cap (fMf)^\omega$  is diffuse. By applying Theorem 6.3 to the subalgebra  $Pf$  of  $fMf$ , we get that either  $(Pf)' \cap (fMf)^\omega \prec_{M^\omega} B^\omega$ ,  $Pf \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or  $Pf_0$  is amenable relative to  $B$ , for some non-zero projection  $f_0 \in \mathcal{Z}(P' \cap M)f$ . Moreover, if  $(Pf)' \cap (fMf)^\omega \prec_{M^\omega} B^\omega$  then since  $(Pf)' \cap (fMf)^\omega$  is diffuse, Lemma 9.5 implies that  $Pf \prec_M B$ .

Altogether, since  $e_0 \leq p$ ,  $f \leq p$  and  $B \subset M_1 \cap M_2$ , we get that either  $Pp \prec_M M_i$ , for some  $i \in \{1, 2\}$ , or  $Pg$  is amenable relative to  $B$ , for some non-zero projection  $g \in \mathcal{Z}(P)p$ . Since  $B$  is amenable, the second condition implies that  $Pp$  has an amenable direct summand, which contradicts the maximality of  $z$ .

In order to finish the proof, assume that  $Pp \prec_M M_i$ , for some  $i \in \{1, 2\}$ . Since  $P' \cap M \subset P$ , it follows that we can find non-zero projections  $p_0 \in Pp$ ,  $q \in M_i$ , a partial isometry  $v \in M$  such that  $v^*v = p_0$  and  $vv^* \leq q$ , and a  $*$ -homomorphism  $\phi : p_0 P p_0 \rightarrow q M_i q$  such that  $\phi(x)v = vx$ , for all  $x \in p_0 P p_0$ . Since  $\phi(p_0 P p_0) \subset q M_i q$  is a diffuse subalgebra and the inclusion  $M_i \subset M$  is mixing, Lemma 9.4 gives that  $\phi(p_0 P p_0)' \cap q M q \subset q M_i q$  and thus  $vv^* \in M_i$ .

Hence, after replacing  $P$  with  $u P u^*$ , for some unitary  $u \in M$ , we may assume that  $p_0 \in M_i$  and  $p_0 P p_0 \subset p_0 M_i p_0$ . Next, we can find a non-zero projection  $p_1 \in p_0 P p_0$  and partial isometries  $v_1, v_2, \dots, v_n \in P$  such that  $v_i^* v_i = p_1$ , for all  $i \in \{1, 2, \dots, n\}$ , and  $p' = \sum_{i=1}^n v_i v_i^*$  is a central projection of  $P$ . Since  $p_1 P p_1 \subset p_1 M_i p_1$ , there exists an embedding  $\theta : P p' \rightarrow \mathbb{M}_n(p_1 M_i p_1)$ .

Since  $M_i$  is strongly solid, [Ho09, Proposition 5.2] gives that  $\mathbb{M}_n(p_1 M_i p_1)$  is also strongly solid. Since the inclusion  $A p' \subset P p'$  is regular and  $A p'$  is a diffuse amenable von Neumann algebra, we deduce that  $P p'$  is amenable. Since  $p' p \neq 0$  (as we have  $0 \neq p_1 \leq p \wedge p'$ ) we again get a contradiction with the maximality of  $z$ . This completes the proof of the theorem.  $\square$

We end with several consequences of Theorem 1.8.

**Corollary 9.6.** *Let  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$  be strongly solid von Neumann algebras.*

*Then  $M = M_1 * M_2$  is strongly solid.*

**Corollary 9.7.** *Let  $(M_1, \tau_1), (M_2, \tau_2), \dots, (M_n, \tau_n)$  be tracial amenable von Neumann algebras with a common von Neumann subalgebra  $B$  such that  $\tau_1|_B = \tau_2|_B = \dots = \tau_n|_B$ . Assume that the inclusions  $B \subset M_1, B \subset M_2, \dots, B \subset M_n$  are mixing. Denote  $M = M_1 *_B M_2 *_B \dots *_B M_n$ .*

*Then  $M$  is strongly solid.*

*Proof.* Since the inclusions  $B \subset M_1, B \subset M_2, \dots, B \subset M_n$  are mixing, it is easy to see that the inclusion  $B \subset M_1 *_B M_2 *_B \dots *_B M_n$  is mixing, for all  $i \in \{1, 2, \dots, n\}$ . The conclusion then follows by using induction and Theorem 1.8.  $\square$

Corollary 9.7 provides two new classes of strongly solid von Neumann algebras.

**Corollary 9.8.** *Let  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$  be countable amenable groups with a common subgroup  $\Lambda$ . Assume that  $g \Lambda g^{-1} \cap \Lambda$  is finite, for every  $g \in (\cup_{i=1}^n \Gamma_i) \setminus \Lambda$ . Denote  $\Gamma = \Gamma_1 *_\Lambda \Gamma_2 *_\Lambda \dots *_\Lambda \Gamma_n$ .*

*Then  $L(\Gamma)$  is strongly solid.*

*Proof.* We claim that the inclusion  $L(\Lambda) \subset L(\Gamma_i)$  is mixing, for every  $i \in \{1, 2, \dots, n\}$ .

To this end, let  $b_n \in (L(\Lambda))_1$  be a sequence converging weakly to 0. We aim to show that  $\|E_{L(\Lambda)}(x b_n y)\|_2 \rightarrow 0$ , for every  $x, y \in L(\Gamma_i) \ominus L(\Lambda)$ . By Kaplansky's density theorem we may assume that  $x = u_h$  and  $y = u_k$ , for some  $h, k \in \Gamma_i \setminus \Lambda$ . Then the set  $F = \{g \in \Lambda \mid h g k \in \Lambda\}$  is finite. Since  $b_n \rightarrow 0$  weakly we get that

$$\|E_{L(\Lambda)}(u_h b_n u_k)\|_2^2 = \sum_{g \in F} |\tau(b_n u_g^*)|^2 \rightarrow 0.$$

Corollary 9.7 now implies that  $L(\Gamma) = L(\Gamma_1) *_L(\Lambda) L(\Gamma_2) *_L(\Lambda) \dots *_L(\Lambda) L(\Gamma_n)$  is strongly solid.  $\square$

Corollary 9.8 generalizes the main result of [Ho09], where the same statement is proven under the additional assumption that for every  $i \in \{1, 2, \dots, n\}$  we can decompose  $\Gamma_i = \Upsilon_i \rtimes \Lambda$ , for some abelian group  $\Upsilon_i$ .

**Corollary 9.9.** *Let  $\Gamma$  be a countable amenable group and  $(D_1, \tau_1), (D_2, \tau_2), \dots, (D_n, \tau_n)$  be tracial amenable von Neumann algebras. Let  $\Gamma \curvearrowright^{\sigma_1} (D_1, \tau_1), \Gamma \curvearrowright^{\sigma_2} (D_2, \tau_2), \dots, \Gamma \curvearrowright^{\sigma_n} (D_n, \tau_n)$  be*

*mixing trace preserving actions. Denote  $D = D_1 * D_2 * \dots * D_n$  and endow  $D$  with its natural trace  $\tau$ . Consider the free product action  $\Gamma \curvearrowright^\sigma (D, \tau)$  given by*

$$\sigma(g)(x_1 x_2 \dots x_n) = \sigma_1(g)(x_1) \sigma_2(g)(x_2) \dots \sigma_n(g)(x_n), \quad \text{for } x_1 \in D_1, x_2 \in D_2, \dots, x_n \in D_n.$$

*Then  $M = D \rtimes \Gamma$  is strongly solid.*

*Proof.* Denote  $M_i = D_i \rtimes \Gamma$ . Since the action  $\Gamma \curvearrowright (D_i, \tau_i)$  is mixing, the inclusion  $L(\Gamma) \subset M_i$  is mixing, for all  $1 \leq i \leq n$ . Since  $\Gamma$  as well as  $D_1, D_2, \dots, D_n$  are amenable, we have that  $M_1, M_2, \dots, M_n$  are amenable. Since  $M = M_1 *_{L(\Gamma)} M_2 * \dots *_{L(\Gamma)} M_n$ , the conclusion follows from Corollary 9.7.  $\square$

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